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Characters and Automorphism Groups of Compact Riemann Surfaces

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Preface

These notes report on recent progress in the determination of automorphism groups of compact Riemann surfaces. Already in the nineteenth century, automorphisms of Riemann surfaces were studied. In this context, an automorphism of a Riemann surface means a biholomorphic self-equivalence, often this is also called a conformal automorphism. In 1893, A. Hurwitz proved in [**Hur93**] that the full automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite, its order being bounded by $84(g-1)$. Two years later, the maximal order of a single automorphism was considered by A. Wiman in [**Wim95**]. More than half a century later, this question was revisited in [**Har66**], this time with emphasis on determining the minimal genus larger than or equal to 2 of a compact Riemann surface admitting an automorphism of prescribed order. The same question was treated for abelian groups in [**Mac65**] and for simple groups, in particular sporadic simple groups, in several publications, e.g. [**Wol89b, Wol89a, CWW92**].

If a group is concretely given as a group of symmetries of a mathematical object and not only abstractly via its isomorphism type, usually results about this group can be refined using representation theoretic methods. A natural representation of the automorphism group of a compact Riemann surface X is that on the complex vector space $\mathcal{H}^1(X)$ of holomorphic abelian differentials on X (see [**FK92**, Chapter V.2]). From this viewpoint, it is quite natural for us to focus our interest on the following two problems.

PROBLEM 1: Classify all groups of automorphisms of compact Riemann surfaces X of fixed genus $g(X) \geq 2$, up to equivalence of the natural action on $\mathcal{H}^1(X)$.

PROBLEM 2: Classify all characters of a given finite group G that arise from the natural action of groups of automorphisms (isomorphic to G) of compact Riemann surfaces X on $\mathcal{H}^1(X)$.

Both problems were treated in several publications by A. Kuribayashi, I. Kuribayashi, and H. Kimura, see [**Kur66, Kur83, Kur84, Kur86, Kur87, KK90a, KK90b, KK91, Kim93**]. We follow their approach and develop it further.

That is, for Problem 1, by the above mentioned bound on the group order, we first note that only finitely many groups must be considered for fixed genus $g \geq 2$. Then we use the fact that a group of automorphisms of a compact Riemann surface of fixed genus occurs as epimorphic image of one of a finite number of finitely presented groups. In the appendix of [**KK90b**], this is expressed by the statement that 'in each genus, there is a finite procedure for deciding which groups arise with $GL_g(\mathbb{C})$-conjugacy from Riemann surfaces'. However, instead of adhoc group theoretic arguments as in [**KK90a, KK90b**], we use a recently obtained classification of groups of small order (see [**BE99**]) or alternatively a recursive construction algorithm to solve Problem 1 for small genera. It should be noted that the classification up to equivalence of the action on $\mathcal{H}^1(X)$ is a refinement of the classification up to isomorphism, and that it means a classification of the characters corresponding to the natural actions on the spaces $\mathcal{H}^1(X)$.

In this sense, Problem 2 is dual to Problem 1. The idea for attacking this problem is to consider certain necessary conditions on a character to arise from an action as required. These conditions are derived from the description of such a character by means of the fixed points of the action on X, a description given by the Eichler Trace Formula. In [**KK91**], it is observed that only very few characters of the general linear group $GL_3(2)$ satisfy these necessary conditions but do *not* arise from natural actions on some $\mathcal{H}^1(X)$. The authors note that 'this phenomenon seems to be rooted in some structure of groups although we cannot explicitly point out which'. We confirm the observation in a more systematic treatment of several classes of groups, including a study of the possibility of infinitely many exceptional characters, as occurs for example in the case of the nonabelian groups of order 8 (see [**Kim93**]).

Each chapter will have its own introductory paragraph. Here we just summarize the contents of the chapters.

In the first three chapters, we introduce briefly the theory that is needed later on. More specifically, **Chapter 1** collects some facts about Riemann surfaces; it is not our aim to give a self-contained introduction to this theory, rather we refer to standard textbooks, e.g. [**Leh64, FK92**], for proofs of classical results. **Chapter 2** first lists the – in fact quite few – needed results in the character theory of finite groups –proofs can be looked up for example in [**Isa76**]. We then deal with the special case we are interested in, the character of the natural action of $G \leq \mathrm{Aut}(X)$ on $\mathcal{H}^1(X)$, for a compact Riemann surface X. **Chapter 3** describes, after studying the example of abelian groups of automorphisms, the relation of this G-character and the fixed points of G on X.

In our situation, a given G-character arises from an action on $\mathcal{H}^1(X)$ for some X if and only if a so-called surface kernel epimorphism onto G exists;

for given G, this is an epimorphism from one of a finite number of finitely presented groups Γ (for details, see Section 3.8). Asking whether such an epimorphism exists means to ask whether a homomorphism exists and if yes whether it is surjective. Both questions are treated in **Chapter 4**, character theoretic arguments such as the interpretation of structure constants and the inspection of the relations between the character tables of a group and its (maximal) subgroups play the main role here. The first section of Chapter 4 lists the necessary tools. They are well-known from problems such as to determine the Hurwitz groups of given order or the strong symmetric genus of a given group (see [**Con85, Con91, CWW92, Tuc83, Wol89b, Wol89a**]), and from related problems concerning the inverse problem of Galois theory (see [**Mat87**]). As an application to Riemann surfaces, the next section shows –for most of the sporadic simple groups as examples– how good bounds for the strong symmetric genus of a group can be computed with character theoretic methods; the library of character tables available in the computer algebra system GAP (see [**S$^+$94**]) is an important source of information for that task. In the last section of Chapter 4, the obtained conditions on the existence of surface kernel epimorphisms are first reformulated in terms of so-called signatures, and then stronger conditions are derived for the case that the group in question is solvable.

With these prerequisites, we are able to present in **Chapter 5** a classification of all groups of automorphisms of compact Riemann surfaces of genus g, as required in Problem 1, for $2 \leq g \leq 48$. In earlier publications, only classifications for $2 \leq g \leq 5$ had been achieved (see [**Kur84, Kur86, KK90a, KK90b**]); an important reason why our methods reach farther than those used there (and in particular farther than those that were available at Hurwitz's time) is that we make heavy use of computer calculations. Chapter 5 describes an algorithm for enumerating the characters in question, and shows several examples. A byproduct is the classification of *irreducible* actions on $\mathcal{H}^1(X)$, the complete list is given in Appendix B.

Then we consider Problem 2, first for the easy case of groups with real character table in **Chapter 6**, and then –motivated by these results– in **Chapter 7** for the general case. The used necessary conditions on a character to be induced by an action of a group of automorphisms of a compact Riemann surface X on the vector space $\mathcal{H}^1(X)$ are formulated as linear equations and inequalities in terms of the decomposition of the given character into irreducibles. These conditions turn out to be sufficient if the multiplicity of the trivial character as a constituent is large enough. For the remaining characters, and hence for the solution of Problem 2 for the given group G, say, the answer depends on G. More specifically, there are groups for which infinite series of characters can be identified that satisfy the necessary conditions but are not induced by the action of a group of automorphisms of a compact

Riemann surface X on $\mathcal{H}^1(X)$; and there are groups for which one can prove that at most finitely many characters satisfying the necessary conditions fail to come from an action on a compact Riemann surface. For example, many finite simple groups are of the latter kind, and the check for this involves only computations with the character table of the given group; if the generic character table of a series of groups is available then this table suffices to perform this check for the finiteness of the set of exceptions for each group in the series. Finally, a full solution of Problem 2 for given (small) groups can be obtained with algorithms that are also presented in Chapters 6 and 7.

Although our arguments are completely elementary, the results are again beyond prior publications. Problem 2 had been considered for the nonabelian groups of order 8 (see [**Kim93**]), all cyclic groups (see [**Kur87**]), and the group $GL_3(2)$ (see [**KK91**]). We generalize this to full classifications for arbitrary dihedral groups (see Section 26), abelian groups (see Section 32), several specific examples such as small symmetric, alternating, and linear groups (see Sections 27.3, 29.3), and partial results for several infinite series of simple and nearly simple groups (see Sections 27.1, 28.1, 29.1, 35.1). Additionally, we simplify the proofs of the known results, mainly due to the concepts introduced in Sections 22 and 30.

A number theoretic problem involving Maillet's Determinant and its generalizations (see [**CO55, Küh79, Tat82**]) arises in Chapter 7. It is studied in Appendix C.

We do *not* consider possible further generalizations, e.g., to study spaces $\mathcal{H}^q(X)$ of differentials of order $q > 1$ (see [**Kur87, FK92**]), automorphism groups of bordered surfaces (see [**BEGG90**]), or higher-dimensional manifolds.

The computational results were obtained with the help of the computer algebra system GAP (see [**S$^+$94**]). A GAP database of the automorphism groups of genera from 2 to 48 is freely available from the author as well as the programs that were used in the computations.

This work was initiated by Prof. Herbert Pahlings. I would like to thank him for his interest in the project. Furthermore I thank Klaus Lux for many discussions. Finally, I owe gratitude to Lehrstuhl D für Mathematik at RWTH Aachen, whose computer equipment I was allowed to use.

Aachen, July 1998 Thomas Breuer

Notation

The standard language and notation of set theory are used, disjoint union of sets A, B is denoted by $A \uplus B$. Besides that, we use $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the ring of integers, and the fields of rational, real, and complex numbers, respectively.

We write ζ_m for the primitive complex m-th root of unity $e^{2\pi i/m}$.

The composition of mappings is written either as $\varphi_2 \circ \varphi_1$, with $(\varphi_2 \circ \varphi_1)(x) = \varphi_2(\varphi_1(x))$, or as $\varphi_1\varphi_2$, with $x(\varphi_1\varphi_2) = (x\varphi_1)\varphi_2 = \varphi_2(\varphi_1(x))$.

G will denote a finite group, $G^\times$ the set of nonidentity elements in G, and G' the derived subgroup of G.

For an element $h \in G$ (note that the letter g will be reserved to denote the genus of compact surfaces), $\langle h \rangle$ is the subgroup of G generated by h, and $|h| = |\langle h \rangle|$ is the *order* of h. $[G{:}H] = |G|/|H|$ is the *index* of the subgroup $H \leq G$ in G, $C_G(H) = \{\sigma \in G \mid \sigma^{-1}h\sigma = h \quad \text{for all} \quad h \in H\}$ the *centralizer* of H in G, $N_G(H) = \{\sigma \in G \mid \sigma^{-1}H\sigma = H\}$ the *normalizer* of H in G. We will write $C_G(h)$ instead of $C_G(\langle h \rangle)$ for an element $h \in G$. A subgroup H is called *normal* if $N_G(H) = G$.

For two groups G and H, $G\colon H$ denotes the split extension of G with H. For example, $\mathbb{Z}^2\colon m$ is a group with a free abelian normal subgroup $\mathbb{Z}^2$ of rank 2 that has a cyclic group of order m as a complement.

Two elements or subgroups H, K of G are *conjugate* if there is $\sigma \in G$ with $\sigma^{-1}H\sigma = K$, we write $H \sim_G K$ in this situation. For a set S on which G acts via conjugation, $S/{\sim_G}$ denotes a set of representatives of the conjugacy classes.

The image of the point x under the group element σ is written as $x\sigma$ if the operation is not conjugation, the *orbit* $\{x\sigma \mid \sigma \in G\}$ of the point x under this action of G is written as xG.

S_n and A_n denote the *symmetric* and *alternating* groups respectively on n symbols, D_{2n} the *dihedral group* of order $2n$, Q_8 the quaternion group of order 8. Finally, $GL_n(F)$, $SL_n(F)$, $PGL_n(F)$, and $PSL_n(F)$ denote the general linear group, the special linear group, the projective general linear group, and the projective special linear group, respectively, in dimension n over the field F (if F is a field) or over a field with F elements (if F is an integer). Instead of $PSL_n(F)$, we also write $L_n(F)$.

For more special notations such as $\mathrm{Fix}_X(h)$ or $I(m)$, the definition in the text can be found via the index at the end of the book.

Names of conjugacy classes and characters refer either to character tables shown in the text or to character tables in [**CCN+85**]; for example, the class 2A of the simple group $L_2(7)$ is the class of involutions, and the classes 7A and 7B are the two classes of elements of order 7, as given in [**CCN+85**, p. 3].