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CHAPTER 1

Compact Riemann Surfaces

For our purposes, it is sufficient to think of Riemann surfaces as topological spaces with additional structure, and this is defined in Section 1.

In Section 2, the necessary terms and facts from algebraic topology are collected. We will need mainly the notions of smooth and branched coverings, universal coverings, and orbit spaces. See [**Mas91**, Chapter V] for a more general and more detailed introduction to these topics. The statements and proofs for the special case that the topological spaces are Riemann surfaces can also be found in [**For77**, Kap. I, §§ 3–5].

We are interested in compact Riemann surfaces of genus at least 2, they arise as orbit spaces $\mathcal{U}/K$ of the upper half plane $\mathcal{U}$ by Fuchsian groups K. These groups are studied in Section 3. The presentation of the material follows [**Leh64**] and [**JS87**].

Some useful statements about the Riemann–Hurwitz Formula can be found in Section 4, and Section 5 sketches two examples.

1. Basic Facts about Riemann Surfaces

Recall that a topological space X is *connected* if it is not the union of two disjoint, nonempty, open sets. X is *Hausdorff* if, for every two distinct points x_1, x_2 in X, there are disjoint neighborhoods of x_1 and x_2. A map $\varphi\colon X \to Y$ is called *continuous* if for each open set $U \subseteq Y$ the preimage $\varphi^{-1}(U)$ is open in X, and φ is called a *homeomorphism* (or *topological equivalence*) if it is continuous and bijective, and its inverse is also continuous.

A connected Hausdorff space X with the property that each point $x \in X$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$ (with its natural topology) is called an n-*manifold*. A 2-manifold is also called a *surface*. By identifying the topological spaces $\mathbb{R}^2$ and $\mathbb{C}$, each point of a surface X has a neighborhood U that is homeomorphic to an open subset V of $\mathbb{C}$.

Such a homeomorphism $\varphi\colon U \to V$ is called a *local coordinate*, and the pair (U, φ) is called a *complex chart* on X. For any two complex charts (U_1, φ_1),

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(U_2, φ_2) with $U_1 \cap U_2 \neq \emptyset$, the coordinate change

$$(\varphi_1^{-1}\varphi_2)|_{\varphi_1(U_1 \cap U_2)} \colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a complex function. It is *holomorphic* (or *analytic* or *regular*) if it is complex differentiable at every point of $\varphi_1(U_1 \cap U_2)$. A holomorphic and bijective map is called *biholomorphic*, its inverse is automatically holomorphic. If $U_1 \cap U_2$ is empty or if the coordinate change is biholomorphic then the charts (U_1, φ_1) and (U_2, φ_2) are called *compatible*.

An *analytic atlas* on X is a system $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$ of pairwise compatible charts with $\bigcup_{i \in I} U_i = X$. Two atlases $\mathcal{A}$, $\mathcal{A}'$ on X are *compatible* if each chart of $\mathcal{A}$ is compatible with each chart of $\mathcal{A}'$, or equivalently if $\mathcal{A} \cup \mathcal{A}'$ is also an atlas on X. Compatibility of analytic atlases is an equivalence relation, the equivalence classes are called *complex structures* on X.

A *Riemann surface* is a surface X together with a complex structure on X.

REMARK 1.1. Some authors do not require Riemann surfaces to be connected (see [**JS87**, pp. 167 ff.]), some authors require Riemann surfaces to be *second countable*, that is, there is a countable basis for its topology (see [**Mir95**, Def. I.1.18]).

Having defined the objects of our interest, we specify the appropriate class of structure preserving maps between Riemann surfaces X and Y as follows. A map $f \colon X \to Y$ is called *holomorphic* if f is continuous, and for each pair of charts $\varphi_1 \colon U_1 \to V_1$ on X and $\varphi_2 \colon U_2 \to V_2$ on Y with $f(U_1) \subseteq U_2$ the map $\varphi_1^{-1} f \varphi_2 \colon V_1 \to V_2$ is holomorphic as a complex function; f is called *biholomorphic* if f is bijective and holomorphic. As for complex functions, the inverse of a biholomorphic map is holomorphic.

If a biholomorphic map $f \colon X \to Y$ exists then X and Y are considered as indistinguishable, and they are called *conformally equivalent* (or *isomorphic as Riemann surfaces*). By definition, conformally equivalent Riemann surfaces are homeomorphic, so the equivalence relation defined by conformal equivalence is a refinement of topological equivalence.

A biholomorphic map $X \to X$ is called a *conformal automorphism* of X. The set of all conformal automorphisms of X is denoted by $\mathrm{Aut}(X)$, it is a group with respect to composition of maps, and is called the *full automorphism group* of X.

REMARK 1.2. Note that the term *automorphism* is used for other kinds of maps in other contexts. For a topological space, any homeomorphism is usually called a *topological automorphism*, and for a covering space (see Section 2), the group of covering transformations is also called the automorphism group of the covering. Besides that, there are of course the obvious notions of group automorphisms, field automorphisms etc., so $\mathrm{Aut}(G)$ denotes the

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automorphism group of the group G. These ambiguities should not cause confusion in the following.

EXAMPLE 1.3. Examples of Riemann surfaces are the complex plane $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the upper half plane $\mathcal{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, and the unit disk $\mathcal{D} = \{z \in \mathbb{C} \mid z \cdot \overline{z} \leq 1\}$. The analytic structure for $\mathbb{C}$, $\mathcal{U}$, and $\mathcal{D}$ is given by the identity map, the one for $\hat{\mathbb{C}}$ can be defined by the two charts

$$\varphi_1 : \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & z \end{array} \quad \text{and} \quad \varphi_2 : \begin{array}{ccc} \hat{\mathbb{C}} \setminus \{0\} & \to & \mathbb{C} \\ z & \mapsto & 1/z. \end{array}$$

The Riemann surfaces $\mathcal{U}$ and $\mathcal{D}$ are conformally equivalent via the holomorphic maps

$$\varphi : \begin{array}{ccc} \mathcal{U} & \to & \mathcal{D} \\ z & \mapsto & (z-\mathrm{i})/(z+\mathrm{i}) \end{array} \quad \text{and} \quad \varphi^{-1} : \begin{array}{ccc} \mathcal{D} & \to & \mathcal{U} \\ z & \mapsto & (-\mathrm{i}z-\mathrm{i})/(z-1), \end{array}$$

see [**JS87**, p. 199]. We can visualize φ as the composition of the rotation $z \mapsto (z-\mathrm{i})/(-\mathrm{i}z+1)$ of the Riemann sphere around the axis through ± 1, mapping 0 to $-\mathrm{i}$, and the rotation $z \mapsto -\mathrm{i}z$ of $\mathcal{D}$ around the origin.

$\mathbb{C}$ and $\mathcal{D}$ are homeomorphic via the map $\mathcal{D} \to \mathbb{C}$, $z \mapsto z/(1-|z|)$, but they are not conformally equivalent because each holomorphic map $\mathbb{C} \to \mathcal{D}$ is bounded and hence constant by Liouville's Theorem [**JS87**, Theorem A.4]. ◇

Every transformation of the form $z \mapsto (az+b)/(cz+d)$ for complex numbers a, b, c, d with $ad-bc=1$ is a conformal automorphism of $\hat{\mathbb{C}}$, and its identification with the pair of matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

defines an isomorphism to the group $PSL_2(\mathbb{C})$. The sets $\mathbb{C}$, $\mathcal{U}$, and $\mathcal{D}$ are subsets of $\hat{\mathbb{C}}$, and the sets of those transformations that fix the point ∞, the circle $\mathbb{R} \cup \infty$, and the unit circle respectively are conformal automorphisms of these Riemann surfaces. Moreover, these are already the full groups of automorphisms.

THEOREM 1.4.

$$\begin{aligned} \mathrm{Aut}(\hat{\mathbb{C}}) &= PSL_2(\mathbb{C}), \\ \mathrm{Aut}(\mathbb{C}) &= \{z \mapsto az+b \mid a, b \in \mathbb{C}, a \neq 0\}, \\ \mathrm{Aut}(\mathcal{U}) &= PSL_2(\mathbb{R}). \end{aligned}$$

Proof. [**JS87**, Theorem 4.17.3] □

Every compact Riemann surface X is homeomorphic to the g-holed torus, for a unique nonnegative integer g, see [**JS87**, Theorem 4.16.1], [**Mir95**,

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Proposition I.1.23], [**For77**, 19.14]. The number g is called the *genus* of X, we will denote it by $g(X)$. For example, we have $g(\hat{\mathbb{C}}) = 0$, and $\hat{\mathbb{C}}$ is the unique compact Riemann surface of genus 0.

2. Basic Algebraic Topology

2.1. Branched Coverings. As above, let X and Y be topological spaces. A subset $A \subseteq X$ is called *discrete* if each point $a \in A$ has a neighborhood V in X such that $V \cap A = \{a\}$. A map $\varphi\colon X \to Y$ is called *discrete* if $\varphi^{-1}(y)$ is a discrete set in X for each $y \in Y$, and φ is called *open* if $\varphi(U)$ is open in Y for each open set $U \subseteq X$.

A continuous, open, and discrete map $\varphi\colon X \to Y$ is called a *branched covering*. In this case, the set $\varphi^{-1}(y)$ is called the *fibre* of $y \in Y$.

The point $x \in X$ is called a *ramification point* if there is no neighborhood U of x such that the restriction $\varphi|_U$ is injective. The image $\varphi(x)$ of a ramification point is called a *branch point*.

A branched covering map is called *unbranched covering* or *smooth covering* if it has no branch points. Smooth coverings can be characterized as local homeomorphisms, in the sense that a map $\varphi\colon X \to Y$ is a smooth covering if and only if each $x \in X$ has a neighborhood U such that $\varphi|_U\colon U \to \varphi(U)$ is a homeomorphism, with $\varphi(U)$ open in Y (see [**For77**, 4.4]).

2.2. Universal Coverings. Let I be the interval $[0,1] \subseteq \mathbb{R}$. A *path* on X is a continuous map $I \to X$. The space X is called *path-connected* (or *arcwise-connected*) if for any two points x_0, $x_1 \in X$ a path u on X with $u(0) = x_0$ and $u(1) = x_1$ exists. A *loop* based at $x_0 \in X$ is a path on X such that $u(0) = u(1) = x_0$. Two loops u, $v\colon I \to X$ based at x_0 are called *homotopic* if there is a continuous map $F\colon I \times I \to X$ such that $F(t,0) = u(t)$ and $F(t,1) = v(t)$ for all $t \in I$, and $F(0,s) = F(1,s) = x_0$ for all $s \in I$.

X is called *simply connected* if X is path-connected and for any $x \in X$, any loop based at x is homotopic to the constant path $I \to X$, $t \mapsto x$. If X is simply connected then a smooth covering $\varphi\colon X \to Y$ has the following universal property (see [**Mas91**, Theorem V.5.1]).

For each connected, smooth covering $\psi\colon Z \to Y$ and each choice of $x_0 \in X$ and $z_0 \in Z$ with $\varphi(x_0) = \psi(z_0)$ there is a unique continuous map $f\colon X \to Z$ with $f(x_0) = z_0$ and $\varphi = f\psi$. Roughly speaking, φ factors through Z.

A smooth covering with this property is called a *universal covering*, if it exists then it is unique up to homeomorphism.

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THEOREM 2.1. *Each Riemann surface X has a simply connected universal covering $\varphi\colon \tilde{X} \to X$, where $\tilde{X}$ is again a Riemann surface and φ is holomorphic.*

The universal covering of any Riemann surface is conformally equivalent to exactly one of $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathcal{U}$.

Proof. [**JS87**, Theorem 4.17.2] □

2.3. Covering Transformations. Let $\varphi\colon X \to Y$ be a smooth covering. A *covering transformation* of φ is a homeomorphism $\sigma\colon X \to X$ with the property that $\sigma\varphi = \varphi$. With composition as multiplication, the covering transformations of φ form a group, which is denoted by $\mathrm{Aut}_Y(X)$. For path-connected X, this group acts fixed point freely on each fibre, see [**Mas91**, V.6.1]. The smooth covering φ is called *regular* (or a *Galois covering*) if $\mathrm{Aut}_Y(X)$ acts transitively on each fibre.

THEOREM 2.2. *A universal covering is regular.*

Proof. [**Mas91**, Lemma V.8.1] □

2.4. Orbit Spaces. For a group G of homeomorphisms of a topological space X, the *canonical projection* onto the *orbit space* X/G is defined to map each point $x \in X$ to its orbit xG under the action of G.

If $X \to Y$ is a smooth covering and $G \leq \mathrm{Aut}_Y(X)$ then the canonical projection $X \to X/G$ is a regular smooth covering. But not all groups of homeomorphisms of X are of this form. So the question is for what groups the projection onto the orbit space has nice properties. We are interested in the case that X is a compact Riemann surface, and the orbit space is to be also a Riemann surface such that the projection is holomorphic.

Such a map is always a finitely branched covering (see [**FK92**, I.2.4]), and a necessary and sufficient condition on a group G that the canonical projection $X \to X/G$ is a finitely branched covering is that G acts *properly discontinuously* on X (see [**FK92**, IV.9]). We will not define here what this means because in the special case $X = \mathcal{U}$, this is equivalent to the condition that the orbits of G on $\mathcal{U}$ are discrete (see [**FK92**, p. 3]). Moreover, by Theorem 1.4, the group $\mathrm{Aut}(\mathcal{U})$ is isomorphic to $PSL_2(\mathbb{R})$, a factor group of $SL_2(\mathbb{R})$ which is a topological space via its natural embedding into $\mathbb{R}^4$. So $\mathrm{Aut}(\mathcal{U})$ can be endowed with the quotient topology and hence is itself a topological space. One can show that the orbits of a subgroup G of $\mathrm{Aut}(\mathcal{U})$ on $\mathcal{U}$ are discrete if and only if G is discrete as a subspace of $\mathrm{Aut}(\mathcal{U})$ (see [**JS87**, Corollary 5.6.4]), thus the following definition describes the groups whose orbit spaces on $\mathcal{U}$ are Riemann surfaces.

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DEFINITION 2.3. A *Fuchsian group* is a discrete subgroup of $\mathrm{Aut}(\mathcal{U})$. A torsion-free Fuchsian group is called a *Fuchsian surface group*.

2.5. Lifts of Automorphisms.

LEMMA 2.4. *Let X be a Riemann surface with universal covering $\pi\colon \tilde{X} \to X$, and $\sigma\colon X \to X$ a (conformal) homeomorphism. Then σ can be lifted to $\tilde{X}$, i.e., there is a (conformal) homeomorphism $\tilde{\sigma}\colon \tilde{X} \to \tilde{X}$ that preserves the fibres of π, i.e., $\tilde{\sigma}\pi = \pi\sigma$.*

Proof. [**For77**, 4.7–4.18], special cases in [**JS87**, 4.18 and 5.9.3]. More general statements hold, see for example [**Acc94**, Theorem 4.11]. □

The condition $\tilde{\sigma}\pi = \pi\sigma$ means that $\tilde{\sigma}$ normalizes the group $\mathrm{Aut}_X(\tilde{X})$ of covering transformations of π. It implies that $\tilde{\sigma}$ is unique up to covering transformations, because for two lifts $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, the quotient $\tilde{\sigma}_1\tilde{\sigma}_2^{-1}$ is a covering transformation of $\tilde{X} \to X$ by

$$(\tilde{\sigma}_1\tilde{\sigma}_2^{-1})\pi = \tilde{\sigma}_1(\tilde{\sigma}_2^{-1}\pi) = \tilde{\sigma}_1\pi\sigma^{-1} = \pi\sigma\sigma^{-1} = \pi.$$

LEMMA 2.5. *For a Riemann surface X with universal covering $\tilde{X}$, we have* $\mathrm{Aut}(X) \cong N_{\mathrm{Aut}(\tilde{X})}(\mathrm{Aut}_X(\tilde{X}))/\mathrm{Aut}_X(\tilde{X})$.

Proof. So the map $\sigma \mapsto \tilde{\sigma}\cdot\mathrm{Aut}_X(\tilde{X})$ defines an injective homomorphism of $\mathrm{Aut}(X)$ into $N_{\mathrm{Aut}(\tilde{X})}(\mathrm{Aut}_X(\tilde{X}))/\mathrm{Aut}_X(\tilde{X})$. In fact it is surjective, since any $\tilde{\sigma} \in N_{\mathrm{Aut}(\tilde{X})}(\mathrm{Aut}_X(\tilde{X}))$ induces a map $\sigma \in \mathrm{Aut}(X)$ via $(x\pi)\sigma = (x\tilde{\sigma})\pi$. □

COROLLARY 2.6. *Suppose $\sigma \in \mathrm{Aut}(X)$ fixes a point $x \in X$. Then one can choose a lift $\tilde{\sigma}$ to the universal covering space $\tilde{X}$ that fixes a point in $\tilde{X}$. Moreover, all lifts of σ are conjugate in $\mathrm{Aut}(\tilde{X})$.*

Proof. (see [**Acc94**, 4.12]) Take $\tilde{x} \in \tilde{X}$ with $\tilde{x}\pi = x$, and $\tilde{\sigma}$ any lift of σ to $\tilde{X}$. Then $\tilde{\sigma}$ respects the fibres of the covering $\pi\colon \tilde{X} \to X$, we have $(\tilde{x}\tilde{\sigma})\pi = x\sigma = x = \tilde{x}\pi$. That is, $\tilde{x}$ and $\tilde{x}\tilde{\sigma}$ lie in the same fibre of π. By Theorem 2.2, there exists a covering transformation $t \in \mathrm{Aut}_X(\tilde{X})$ with $\tilde{x}(\tilde{\sigma}t) = \tilde{x}$. So $\tilde{\sigma}t$ is a lift of σ that fixes $\tilde{x}$.

Let $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ be two lifts of σ that fix the points $\tilde{x}_1$ and $\tilde{x}_2$, respectively. Again by Theorem 2.2, there is $t \in \mathrm{Aut}_X(\tilde{X})$ with $\tilde{x}_1 t = \tilde{x}_2$, and $t\tilde{\sigma}_2 t^{-1}$ fixes $\tilde{x}_1$. But then $t\tilde{\sigma}_2 t^{-1} = \tilde{\sigma}_1$ because the quotient of the elements on the two sides is a covering transformation that fixes a point, i.e., it is the identity on $\mathcal{U}$. □

REMARK 2.7. A group theoretic interpretation of the above statement about lifts of automorphisms with fixed points is that $\langle\tilde{\sigma}\rangle$ is a complement of

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$\mathrm{Aut}_X(\tilde{X})$ in the extension $\langle \mathrm{Aut}_X(\tilde{X}), \tilde{\sigma} \rangle$, and that all such complements are conjugate.

3. Fuchsian Groups

A Riemann surface X will be parametrized as the orbit space of its universal covering space $\tilde{X}$, under the action of the subgroup $\mathrm{Aut}_X(\tilde{X})$ of $\mathrm{Aut}(\tilde{X})$. We are interested in compact Riemann surfaces of genus at least 2, in all these cases we have $\tilde{X} = \mathcal{U}$, the upper half plane (see [**JS87**, Theorem 4.19.8]). Before we look more closely at the groups that occur as $\mathrm{Aut}_X(\mathcal{U})$, we collect some facts about the action of $\mathrm{Aut}(\mathcal{U})$ on $\mathcal{U}$.

3.1. Hyperbolic Distance and Area. For a piecewise differentiable path $\gamma\colon [0,1] \to \mathcal{U}$ with $\gamma(t) = x(t) + \mathrm{i}y(t)$, we set

$$h(\gamma) = \int_0^1 \frac{1}{y} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and define the *hyperbolic distance* ρ on $\mathcal{U}$ by

$$\rho(z, w) = \inf_{\gamma} \{h(\gamma)\}$$

where the infimum is taken over all piecewise differentiable paths $\gamma\colon [0,1] \to \mathcal{U}$ with $\gamma(0) = z$ and $\gamma(1) = w$. Together with ρ, $\mathcal{U}$ is a metric space. Moreover, $\mathrm{Aut}(\mathcal{U})$ acts as a group of *isometries* with respect to ρ.

In the definition of $\rho(z, w)$, the infimum is in fact a minimum, and there is a unique path of shortest hyperbolic length that joins z and w, which is called an *H-line segment.*

THEOREM 3.1. *The H-line segments in $\mathcal{U}$ are arcs of semi-circles with center on the real axis or segments of Euclidean lines perpendicular to the real axis.*

Proof. [**JS87**, Theorem 5.3.3] □

The *hyperbolic area* of a subset $E \subseteq \mathcal{U}$ is defined as

$$\mu(E) = \iint_E \frac{dxdy}{y^2}$$

if this integral exists. Then $\mu(E)$ is invariant under all transformations in $\mathrm{Aut}(\mathcal{U})$.

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3.2. A Presentation for Fuchsian Groups.

THEOREM 3.2. *If Γ is a Fuchsian group with compact orbit space $\mathcal{U}/\Gamma$ of genus g then there are elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots, c_r$ in* $\mathrm{Aut}(\mathcal{U})$ *such that the following hold.*

1. *We have $\Gamma = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots, c_r \rangle$.*
2. *Defining relations for Γ are given by*

$$c_1^{m_1}, c_2^{m_2}, \ldots, c_r^{m_r}, \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} c_j,$$

   where the m_i are integers with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$.
3. *Each nonidentity element of finite order in Γ lies in a unique conjugate of $\langle c_i \rangle$ for suitable i. Furthermore, the cyclic groups $\langle c_i \rangle$ are self-normalizing in Γ.*
4. *Each nonidentity element of finite order in Γ (a so-called* elliptic *element) has a unique fixed point in $\mathcal{U}$. Each element of infinite order in Γ (a so-called* hyperbolic *element) acts fixed point freely on $\mathcal{U}$.*

Proof. See [**Leh64**, p. 227 and p. 234] or [**Sah69**, Appendix]. □

3.3. Signatures. For a Fuchsian group Γ as in Theorem 3.2, the numbers g, r, and m_1, m_2, ..., m_r are uniquely determined; note that $2g$ is the rank of the free abelian part of the commutator factor group of Γ, see Lemma A.3. The following lemma shows that the ordering of the values m_i does not impose any extra condition.

LEMMA 3.3. *Let G be a group, and c_1, c_2, ... $c_r \in G$. For each permutation π of $\{1, 2, \ldots, r\}$, there are elements h_1, h_2, ..., $h_r \in G$ such that $\prod_{i=1}^{r}(h_i c_{\pi(i)} h_i^{-1})$ is G-conjugate to $\prod_{i=1}^{r} c_i$.*

Proof. Without loss of generality assume that $\pi = (i, i+1)$, with $1 \leq i \leq r-1$, and observe that $c_i c_{i+1} = (c_i c_{i+1} c_i^{-1}) c_i$. □

We call $(g; m_1, m_2, \ldots, m_r)$ the *signature* of Γ, the integer g is called its *orbit genus*, and the m_i are called the *periods* of Γ.

As a consequence of Theorem 3.2, the isomorphism type of Γ is determined by the signature of Γ.

REMARK 3.4. In the literature, more general signatures are also considered that may have infinite periods, the signatures we have introduced are then called *finite signatures*, see for example [**Sah69**, Appendix]. The orbit spaces of groups with infinite signatures are not compact, so we are only interested in Fuchsian groups with finite signatures.

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3.4. Groups of Genera 0 and 1. Compact Riemann surfaces of genus 0 and 1 and finite subgroups of their –infinite– automorphism groups can also be uniformized by groups Γ with finite signatures and torsion-free normal subgroups $K \leq \Gamma$. Since the universal covering spaces of these Riemann surfaces are $\hat{\mathbb{C}}$ and $\mathbb{C}$, respectively, the groups Γ are subgroups of $\mathrm{Aut}(\hat{\mathbb{C}})$ and $\mathrm{Aut}(\mathbb{C})$.

The possible signatures for Γ in the case that K has orbit genus 0 satisfy $-1+\frac{1}{2}\sum_{i=1}^{r}(1-1/m_i) < 0$, K is trivial, and the groups Γ are finite polyhedral groups; they are exactly the finite subgroups of the infinite group $\mathrm{Aut}(\hat{\mathbb{C}})$. Table 1 lists the signatures and the isomorphism types of these groups. Note that the signature of a polyhedral group is uniquely determined only if the periods are forced to be element orders in the group in question; thus we exclude in the following "signatures" $(0; m)$, with arbitrary $m \geq 2$, for the trivial group and $(0; m_1, m_2)$, with $m_1 \neq m_2$, for the cyclic group of order $\gcd(m_1, m_2)$.

$(g_0; m_1, \ldots, m_r)$	Γ	$\lvert\Gamma\rvert$
$(0; -)$	1	1
$(0; n, n)$	n	n
$(0; 2, 2, n)$	D_{2n}	$2n$
$(0; 2, 3, 3)$	A_4	12
$(0; 2, 3, 4)$	S_4	24
$(0; 2, 3, 5)$	A_5	60

TABLE 1. Signatures and Groups in Genus 0

The possible signatures for Γ in the case that K has orbit genus 1 satisfy $-1+\frac{1}{2}\sum_{i=1}^{r}(1-1/m_i) = 0$. Table 2 lists the signatures and the isomorphism types of the groups. The full automorphism group of any Riemann surface of genus 1 is infinite, and isomorphic to one of $(\mathbb{C}/\mathbb{Z}^2)\colon 2$, $(\mathbb{C}/\mathbb{Z}^2)\colon 4$, and $(\mathbb{C}/\mathbb{Z}^2)\colon 6$.

$(g_0; m_1, \ldots, m_r)$	Γ
$(1; -)$	$\mathbb{Z}^2$
$(0; 2, 3, 6)$	$\mathbb{Z}^2\colon 6$
$(0; 2, 4, 4)$	$\mathbb{Z}^2\colon 4$
$(0; 3, 3, 3)$	$\mathbb{Z}^2\colon 3$
$(0; 2, 2, 2, 2)$	$\mathbb{Z}^2\colon 2$

TABLE 2. Signatures and Groups in Genus 1

For more details, see [**Sah69**, Appendix], [**JS87**, Theorem 2.13.5] or [**FK92**, IV.9.3].

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3.5. Fundamental Regions. Each Fuchsian group Γ has a *fundamental region* in $\mathcal{U}$, that is, a closed subset $F \subseteq \mathcal{U}$ with the properties that $\bigcup_{\sigma \in \Gamma} F\sigma = \mathcal{U}$ and that the intersection $F \cap F\sigma$ is contained in the boundary of F for all $\sigma \in \Gamma$. In particular, if $p \in \mathcal{U}$ is a point with trivial stabilizer in Γ then

$$D_p(\Gamma) = \{z \in \mathcal{U} \mid \rho(z,p) \leq \rho(z,p\gamma) \quad \text{for all} \quad \gamma \in \Gamma\}$$

is a connected fundamental region for Γ, the so-called *Dirichlet region for* Γ *centered at* p (see [**JS87**, Theorem 5.8.3], for the existence of points with trivial stabilizer see [**JS87**, Theorem 5.6.3 (ii)]). If $\mathcal{U}/\Gamma$ is compact then the boundary of a Dirichlet region is a finite number of H-line segments and thus has zero hyperbolic area.

THEOREM 3.5. *If F is a fundamental region whose boundary has zero hyperbolic area for a Fuchsian group with signature $(g; m_1, m_2, \ldots, m_r)$ then we have*

$$\mu(F) = 2\pi \left((2g-2) + \sum_{i=1}^{r} (1 - 1/m_i) \right).$$

Proof. [**JS87**, Theorem 5.10.3] □

Let F be a Dirichlet region for Γ, and s a side of F, i.e., one of the hyperbolic arcs that form the boundary of F. If the image $s\gamma$ of s under $\gamma \in \Gamma^{\times}$ is also a side of F then $s\gamma$ is called the *conjugate side* of s. One can show that the boundary of F consists of conjugate pairs of sides (see [**JS87**, pp. 245–247]).

If a side s is self-conjugate then the conjugating transformation fixes the point in the middle of s. In this case, we count each half of s as a side of its own, and consider the point in the middle as a vertex of F.

The vertices of F that are points with nontrivial stabilizer in Γ are called *elliptic* vertices, the others are called *accidental*. Note that points with nontrivial stabilizer *must* be vertices of each fundamental region that contains them, whereas the position of accidental vertices depends on the fundamental region, e.g., on the choice of the center of a Dirichlet region.

It may happen that several vertices of a fundamental region lie in the same orbit of Γ, but we may choose the fundamental region such that no two elliptic vertices are congruent modulo Γ. The boundary of such a fundamental region is called a *canonical polygon*. Each pair of conjugate sides meets in an elliptic vertex of a canonical polygon.

If Γ has r conjugacy classes of maximally cyclic subgroups and orbit genus 0 then F has $2r$ sides and $2r$ vertices, r accidental vertices and r elliptic vertices, one for each conjugacy class of elliptic generators of Γ. In a walk around a canonical polygon for Γ, elliptic and accidental vertices alter. If the

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