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Auslander-Buchweitz Approximations of Equivariant Modules

Mitsuyasu Hashimoto
Nagoya University

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Introduction

Let R be a commutative ring, and G an affine flat group scheme over R. We say that A is a (commutative) G-algebra if A is a G-module and is a (commutative) R-algebra, and the product $A \otimes A \to A$ is G-linear. We say that M is a (G, A)-module (or G-equivariant A-module) if M is an A-module and is a G-module, and the A-action $A \otimes M \to M$ is G-linear. A (G, A)-linear map simply means a G-linear A-linear map. Thus, we get an abelian category ${}_{G,A}\mathbb{M}$ with enough injectives. The main purpose of these notes is to discuss homological aspects of (G, A)-modules, from the viewpoint of commutative ring theory of R and A.

In particular, we study various (weak) Auslander–Buchweitz contexts which appear there. The theory of Cohen–Macaulay approximations over Cohen–Macaulay local rings by Auslander and Buchweitz [10] contributes greatly to the new developments in commutative ring theory [148]. On the other hand, their theory of approximations is given in rather general form as a theory of abelian categories [10, 11], and its applications are appearing in so many topics of algebras. (Weak) Auslander–Buchweitz contexts (I.1.12) are one of its formulations.

Auslander and Reiten [11] proved that, in the category of finite modules over a finite dimensional algebra over a field, Auslander–Buchweitz contexts and basic cotilting modules are in one-to-one correspondence. Miyachi [112] proved that we have an Auslander–Buchweitz context from a cotilting module in a rather general situation. Cohen–Macaulay approximations over Cohen–Macaulay local rings with canonical modules are a special case.

Moreover, as an application of the Auslander–Reiten correspondence above, C. M. Ringel proved the existence of Δ-good approximations over a quasi-hereditary algebra [131]. From Ringel's theorem, the existence of Δ-good approximations of finite dimensional representations of reductive groups immediately follows, using Schur algebras. Quasi-hereditary algebras were originally introduced by Cline–Parshall–Scott [36, 134] to study the representation categories of reductive groups. S. Donkin studied tilting modules of reductive groups [50], and this direction is developing. Note that a tilting module of a reductive group, which is one of the important consequences of Ringel's approximations, is the cotilting module corresponding to Ringel's Auslander–Buchweitz context over a Schur algebra by the Auslander–Reiten correspondence. It is also a tilting module in the sense of [113], and thus the name 'tilting' is used for both concepts.

Our goal is to construct (weak) Auslander–Buchweitz contexts in the category of A-finite (G, A)-modules when G is a split reductive group over a commutative noetherian ring R, and A is a commutative noetherian G-algebra. These approximations generalize Cohen–Macaulay approximations in commutative ring theory and Ringel's approximations in the representa-

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tion theory of algebraic groups simultaneously.

Some applications are also included. As an application of the construction of Ext_A-groups as equivariant modules, we prove a new criterion of Cohen–Macaulay, Gorenstein, local complete intersection, and regular properties of noetherian G-algebras (Matijevic–Roberts type theorem). The case where G is a split torus is known as a criterion for multi-graded rings by Goto–Watanabe [60]. Homological theory for graded modules over graded rings by Goto–Watanabe [59, 60] is a strong motivation for the study of homological theory of equivariant modules here.

There are applications of the Auslander–Buchweitz theory of equivariant modules, too. The first one is an application to the problem of resolutions of determinantal rings. We prove that there is a (non-minimal) equivariant resolution of a determinantal ring with the same length as the minimal free resolution, and with each term a direct summand of a finite direct sum of tensor products of exterior powers. For the maximal minor case, Buchsbaum [28] constructed such a resolution explicitly.

The second one is an application to invariant theory. Let G be a reductive group over a perfect field of positive characteristic, V a finite dimensional G-module, and assume that the symmetric algebra $S := \mathrm{Sym}\, V$ admits a good filtration as a G-module. Then S^G is strongly F-regular [76]. This result yields a similar result over an arbitrary base ring, using (u-)good modules over an arbitrary base discussed here (in preparation, and we omit it).

Other related results on integral representations, most of which are necessary for Auslander–Buchweitz theory, are included. Highest weight theory for coalgebras over an arbitrary base, including the theory of Schur algebras and tilting modules, is also new here. Note that representations and Schur algebras over a Dedekind ring are treated in [48, 51], and quasi-hereditary algebras over an arbitrary base are treated in [38]. Our approach is slightly different, and is a natural extension of the original approach for the field base due to Donkin in [48].

In Chapter I, we introduce basic notions from each algebraic topic used later. Namely, from homological algebra, commutative ring theory, theory of schemes and sheaves, Hopf algebra theory, and representation theory of algebraic groups and algebras. We have devoted considerable space to this chapter. This is partly to make the notes accessible to a wider range of readers interested in algebra, including graduate students, and partly to lay a foundation for the homological theory of Hopf algebras over an arbitrary base ring. We only assume that readers are familiar with the elements of each theory. In general, the results in the chapter are preliminary results and/or well-known, and most of the known results are listed without proofs but with references as far as possible.

Some generalities are included only for record. For example, (I.5.1) and (I.5.2) will not be used later very much. Similarly for the summary on exact

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categories and their derived categories in (I.1.3)–(I.1.6), and the reader may skip these subsections in the first reading, provided he or she is familiar with the homological algebra of the usual abelian categories (the differences from the usual homological algebra will be used only in (I.3.6)).

A G-module is nothing but an $R[G]$-comodule. A (G, A)-module is nothing but an $(R[G], A)$-Hopf module. For future reference, we have included a long section on elementary homological algebras of comodules and Hopf modules (I.3).

In chapter II, we construct Ext_A and Tor^A functors of (G, A)-modules. When G is the split torus $\mathbb{G}_m^n$, then to say that A is a G-algebra is the same as to say that A is $\mathbb{Z}^n$-graded. Similarly, a (G, A)-module is nothing but a graded A-module. Homological algebra of $\mathbb{Z}^n$-graded rings and modules was studied by Goto–Watanabe [59, 60] from the viewpoint of commutative ring theory. In their study, the facts that if M and N are graded A-modules then $\operatorname{Tor}_i^A(M, N)$ is again graded in a natural way, and that if A is noetherian and M is A-finite then $\operatorname{Ext}_A^i(M, N)$ is also graded in a natural way, are used effectively. The purpose here is to generalize these facts to more general G. Moreover, some results on rings and modules are generalized to those of quasi-coherent sheaves over schemes in (II.2.3).

In (II.2.4), we give an application of the results in chapter II. Namely, we generalize a result on homological properties (Cohen–Macaulay, Gorenstein, l.c.i., and regular) for graded rings by Matijevic–Roberts and others. Moreover, in (II.2.2) we give a sufficient condition for the category of G-modules to be a full subcategory of the module category of the hyperalgebra of G. We also discuss the projectivity of the coordinate ring in (II.2.2) and give a new criterion.

In chapter III, we generalize Ringel's approximation for quasi-hereditary algebras, which is an important example of an Auslander–Buchweitz context, to those for some general coalgebras over an arbitrary noetherian commutative ring R. We discuss a variation (or a coalgebra version) of the theory by Cline–Parshall–Scott and Donkin (over a field), before we consider a general base ring.

All the results in this chapter are about G-modules (or C-comodules), and the commutative algebra A (on which G or C acts or coacts) does not appear. However, as a G-module is an R-module, ring theoretic properties of R and module theoretic properties of modules over R play important roles here. Although projectivity of (non-finite) modules is usually less important than flatness in algebraic geometry, the projectivity of the coalgebra C (as an R-module) is very important here, as in Seshadri's important paper [135].

We review the theory of Cline–Parshall–Scott and Donkin, and this part (III.1) is reasonably self-contained. Everything is done in the coalgebra language here, for later generalization to the case where the base ring is arbitrary.

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The notion of good modules (or modules with good filtrations) is divided into two, u-good modules and good modules. U-goodness, which is an abbreviation for universal goodness, means goodness which is stable under base change. A good module is not u-good in general. However, we prove that an R-finite R-projective good module is u-good.

In (III.4.4), we construct an Auslander–Buchweitz context which generalizes both Ringel's approximation and Cohen–Macaulay approximation simultaneously. A generalization of Sharp's theorem by Avramov–Foxby (Theorem I.4.10.19) is important here.

In chapter IV, we construct an Auslander–Buchweitz context in the category of (G, A)-modules. We only consider positively graded A and graded (G, A)-modules M, with each homogeneous component of A or M a G-submodule. Such a situation is realized on replacing G by another reductive group, see (IV.1.1). As in chapter III, we have two (weak) Auslander–Buchweitz contexts corresponding to regular and Cohen–Macaulay properties. The results are new, even if we assume that the base ring R is a field. In (IV.2), as an example and an application, we study resolutions of determinantal rings.

Determinantal rings have always been interesting examples in commutative ring theory. Subsection (IV.2.1) is a survey of the problem of resolutions of determinantal rings. Applications of the equivariant method have been successful in studying determinantal rings. A. Lascoux and Pragacz–Weyman determined the equivariant minimal free resolution of $A = S/I_t$ as a graded S-module in characteristic zero. In particular, the Betti numbers $\dim_k \operatorname{Tor}_i^S(k, A)$ were determined with the determination of the irreducible decomposition of $\operatorname{Tor}_i^S(k, A)$ as representations. After Lascoux's work, the equivariant method has been used to study syzygies of determinantal rings over an arbitrary base ring (see [77]).

As an application of the construction of an Auslander–Buchweitz context in the category of (G, S)-modules, we show that there is an equivariant finite free resolution of S/I_t whose length is equal to $\operatorname{proj.dim}_S S/I_t$ and each term of which consists of $S \otimes T$, with T tilting. This gives a partial answer to the question posed by D. A. Buchsbaum and J. Weyman, and generalizes the generalized Koszul complex by Buchsbaum.

Acknowledgement. These notes are widely extended versions of the informal notes ([74], in Japanese) for the author's lectures at Tokyo Metropolitan University in 1996. Many of the results proved here have been announced in [75]. I am grateful to my distinguished friend Professor Kazuhiko Kurano for providing the opportunity to give the lecture, and to write the notes. He also kindly helped me in error-corrections of the draft. Special thanks are also due to Professor Luchezar L. Avramov, Professor David A. Buchsbaum, Professor Yukio Doi, Professor David Eisenbud, Professor Shiro Goto, Professor Masaharu Kaneda, Professor Jun-ichi Miyachi, Professor Masayoshi

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Conventions and terminology

An equation of the form $A := B$ reads 'A is defined to be B.'

For a set X, $\#X$ stands for the cardinality of X. The symbols $\mathbb{N}$ and $\mathbb{N}_0$ respectively stand for the set of positive and non-negative integers. Semigroups and rings are always required to have unit elements, and semigroup homomorphisms and ring homomorphisms are always required to preserve unit elements. Subsemigroups and subrings are required to have the unit elements in common. Unit elements are required to act as identities for semigroup actions on sets and ring actions on additive groups. For a semigroup G, we denote by $G^\times$ the group of invertible elements of G. For a ring A, this notation applies to the multiplicative semigroup A, and hence $A^\times$ is the unit group of A.

Throughout these notes, the symbol R always stands for a commutative ring. The symbols $\otimes$ and Hom stand for $\otimes_R$ and Hom_R, respectively. If the ring R in question happens to be a field, then we sometimes let $R = k$, and use k. In this case, $\otimes$ and Hom stand for $\otimes_k$ and Hom_k, respectively.

The word 'scheme' always means a separated scheme. For R-schemes X and Y, $X(Y)$ stands for the set of R-morphisms from Y to X. For a scheme X and an X-scheme Y, we say that Y is algebraic over X if Y is of finite type over X. We say that Y is a *variety* over X if Y is algebraic over X and Y is an integral scheme. If Y is a closed subscheme of X and is integral, then we say that Y is a *subvariety* of X. A *geometric point* of an R-scheme X is an algebraically closed field K which is an R-algebra, together with an R-morphism $\operatorname{Spec} K \to X$. If R is not specified, then we assume $R = \mathbb{Z}$. A geometric fiber of an R-morphism $Y \to X$ is the fiber $Y \times_X \operatorname{Spec} K$ for some geometric point $\operatorname{Spec} K \to X$. For an R-scheme X and a commutative R-algebra R', the base change $\operatorname{Spec} R' \times_{\operatorname{Spec} R} X$ is sometimes denoted by $R' \otimes X$.

For a scheme X, an *$\mathcal{O}_X$-module* means an $\mathcal{O}_X$-module sheaf. If we want to mean a presheaf, then we call it an $\mathcal{O}_X$-module presheaf. The abelian category of $\mathcal{O}_X$-modules is denoted by ${}_X\mathbb{M}$. The full subcategory of quasi-coherent (resp. coherent) $\mathcal{O}_X$-modules is denoted by $\mathrm{Qco}(X)$ (resp. $\mathrm{Coh}(X)$).

For a commutative ring A, $\operatorname{Spec} A$ denotes the prime spectrum of A (and the set of prime ideals of A). For an A-module M and $\mathfrak{p} \in \operatorname{Spec} A$, $M_\mathfrak{p}$ denotes the localization of M with respect to the multiplicatively closed

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subset $A \setminus \mathfrak{p}$ of A. The residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ of the local ring $A_\mathfrak{p}$ is denoted by $\kappa(\mathfrak{p})$. For an A-module M, $M(\mathfrak{p})$ stands for the $\kappa(\mathfrak{p})$-vector space $\kappa(\mathfrak{p}) \otimes_A M$. For an A-scheme X, the fiber $\kappa(\mathfrak{p}) \otimes_A X$ is denoted by $X(\mathfrak{p})$. The symbol $\operatorname{Max} A$ stands for the set of maximal ideals of A. For an A-module M, $\operatorname{supp} M$ stands for the support $\{\mathfrak{p} \in \operatorname{Spec} A \mid M_\mathfrak{p} \neq 0\}$ of M. A minimal element of $\operatorname{supp} M$ (with respect to the incidence relation) is called a *minimal prime* of M, and the set of minimal primes of M is denoted by $\operatorname{Min} M$. The set of associated primes of M is denoted by $\operatorname{Ass} M$. A finite A-module means a finitely generated A-module. For an A-module M, the corresponding quasi-coherent sheaf over $\operatorname{Spec} A$ is denoted by $\tilde{M}$.

Let A be a local ring with the unique maximal ideal $\mathfrak{m}$. We express this situation by saying that $(A, \mathfrak{m})$ is a local ring.

For a scheme X, $x \in X$ and an abelian presheaf $\mathcal{M}$ over X, $\mathcal{M}_x$ denotes the stalk of $\mathcal{M}$ at x. The maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_{X,x}$ or sometimes by $\mathfrak{m}_x$, and the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is denoted by $\kappa(x)$.

If $f : X \to \operatorname{Spec} R$ is an R-scheme, M is an R-module, and $\mathcal{N} \in {}_X\mathbb{M}$, then we denote $f^*\tilde{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ by $M \otimes \mathcal{N}$.

For an affine R-scheme $X = \operatorname{Spec} A$, we sometimes denote its coordinate ring $A = \Gamma(X, \mathcal{O}_X)$ by $R[X]$.

For a ring A, an A-module means a left A-module, unless otherwise specified. The category of A-modules is denoted by ${}_A\mathbb{M}$. The opposite ring of A is denoted by A^{op}. The category of right A-modules is denoted by $\mathbb{M}_A$ or ${}_{A^{\mathrm{op}}}\mathbb{M}$. For an A-module M, we denote the injective hull (injective envelope) of M by $E_A(M)$.

Some remarks on terminology in category theory are collected in Remark I.1.12.15.

These notes consist of four chapters I – IV, and each chapter consists of several sections, and each section is divided into several subsections. Theorem, paragraph and equation numbering are unified, and any number is of the form `sec.subsec.thm(par,eqn)num`.

Cross references in these notes are in the form

`chap.sec.subsec.thm(par,eqn)`,

where `chap` denotes the chapter number (in a capital Roman numeral), and `sec`, `subsec`, `thm`, `par`, `eqn` respectively denote the section, subsection, theorem, paragraph, and the equation number (in Arabic). However, for references from the same chapter, the chapter number is omitted. A reference with only two Arabic numerals (preceded by a chapter number if any) such as (IV.2.2) shows the `[chap.]sec.subsec` number, and refers to the corresponding whole subsection. At the beginning (or sometimes in the middle) of each subsection or paragraph, any general assumption effective throughout the subsection or paragraph is shown, and usually such assumptions are not shown again in each theorem (proposition or lemma) therein.

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