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Chapter I

Background Materials

1 From homological algebra

1.1 Yoneda's lemma

(1.1.1) We denote the category of sets by Set, the category of groups by Grp, and the category of abelian groups by Ab or by ${}_{\mathbb{Z}}\mathbb{M}$. If $\mathcal{C}$ is a category, then we denote the set of objects of $\mathcal{C}$ by $\mathrm{ob}(\mathcal{C})$. However, by abuse of notation, we sometimes denote it also by $\mathcal{C}$. For $M, N \in \mathcal{C}$, the set of morphisms from M to N in $\mathcal{C}$ is denoted by $\mathcal{C}(M, N)$ or $\mathrm{Hom}_{\mathcal{C}}(M, N)$. The opposite category of $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$.

(1.1.2) For categories $\mathcal{C}$, $\mathcal{C}'$, a functor $F : \mathcal{C} \to \mathcal{C}'$ and a full subcategory $\mathcal{S}$ of $\mathcal{C}$, we denote the full subcategory of $\mathcal{C}'$ consisting of objects isomorphic to some $F(S)$ for $S \in \mathcal{S}$ by $F(\mathcal{S})$. For a category $\mathcal{C}$, we denote both a single null object of $\mathcal{C}$ and the set of all null objects by the same symbol 0, which will not cause confusion. For $C \in \mathcal{C}$, the category of objects over C is denoted by $\mathcal{C}/C$. An object of $\mathcal{C}/C$ is a morphism $\varphi : D \to C$ with the codomain C, and a morphism from $\varphi : D \to C$ to $\psi : E \to C$ is a morphism $f : D \to E$ of $\mathcal{C}$ such that $\psi f = \varphi$. The composition of morphisms of $\mathcal{C}/C$ is the same as that of $\mathcal{C}$.

(1.1.3) Let $\mathcal{U}$ be a *universe* (see, e.g., [9]) and $\mathcal{C}$ a category. We say that $\mathcal{C}$ is a $\mathcal{U}$-category if $\mathcal{C}(C, C') \in \mathcal{U}$ for objects C, C' of $\mathcal{C}$. We fix once for all a universe $\mathcal{U}$, and we only consider this universe $\mathcal{U}$ unless otherwise specified. An element of $\mathcal{U}$ is sometimes referred as a *small set*. Unless otherwise specified, a category means a $\mathcal{U}$-category. However, when we form a new category from some $\mathcal{U}$-categories, exceptions (see (1.1.5) and (1.4.7)) may apply. If we need to emphasize a category is a $\mathcal{U}$-category, we use the expression 'category with small hom sets.'

The category Set means the category of elements of $\mathcal{U}$, and the category Grp means the category of small groups (i.e., groups in $\mathcal{U}$), and so on, and these concrete categories are also $\mathcal{U}$-categories.

We say that a category $\mathcal{C}$ is *small* if $\mathrm{ob}(\mathcal{C}) \in \mathcal{U}$. We say that a category $\mathcal{C}$ is *svelte* (or *skeletally small*) if there exists a small full subcategory $\mathcal{D}$ of $\mathcal{C}$ such that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is an equivalence.

(1.1.4) Let $\mathcal{C}$ be a category, and R a commutative ring. We say that $\mathcal{C}$ is an *R-category* if $\mathcal{C}(M,N)$ is an R-module for $M,N \in \mathcal{C}$, and the composition $\mathcal{C}(M,N) \times \mathcal{C}(L,M) \to \mathcal{C}(L,N)$ is R-bilinear. A $\mathbb{Z}$-category is also called an $\underline{\mathrm{Ab}}$-category, or a *preadditive category*. A preadditive category with finite direct products (in particular, with a terminal object) is called an *additive category*. A finite direct product in an additive category is naturally isomorphic to the coproduct, and in particular, the category has a null object.

Let $\mathcal{A}$ and $\mathcal{B}$ be R-categories, and $F : \mathcal{A} \to \mathcal{B}$ a functor. We say that F is an *R-linear functor* if the canonical map

$$F : \mathrm{Hom}_{\mathcal{A}}(a,b) \to \mathrm{Hom}_{\mathcal{B}}(Fa,Fb)$$

is R-linear for any $a,b \in \mathcal{A}$. A $\mathbb{Z}$-linear functor is called an *additive functor*.

(1.1.5) For categories $\mathcal{A}$ and $\mathcal{B}$, we denote the set of functors from $\mathcal{A}$ to $\mathcal{B}$ by $\mathrm{Func}(\mathcal{A},\mathcal{B})$. For $F,G \in \mathrm{Func}(\mathcal{A},\mathcal{B})$, we denote the set of natural transformations from F to G by $\mathrm{Nat}(F,G)$. Note that $\mathrm{Func}(\mathcal{A},\mathcal{B})$ is a (not necessarily small) category with $\mathrm{Nat}(F,G)$ its hom set.

For $A \in \mathcal{A}$, when we define $y(A) := \mathrm{Hom}_{\mathcal{A}}(?,A)$, we get a functor $y : \mathcal{A} \to \mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \underline{\mathrm{Set}})$. The following is well-known as Yoneda's lemma.

Lemma 1.1.6 *Let $\mathcal{A}$ be a category, and $T : \mathcal{A}^{\mathrm{op}} \to \underline{\mathrm{Set}}$ a functor. Then we have that the natural map $Y : T \to \mathrm{Nat}(y(?),T)$ given by*

$$(Y(t))(\varphi) = (T(\varphi))(t) \quad (\text{for } A,B \in \mathrm{ob}(\mathcal{A}),\ t \in T(A),\ \varphi \in \mathcal{A}(B,A))$$

is an isomorphism, whose inverse is given by

$$\mathrm{Nat}(y(A),T) \to T(A) \qquad (f \mapsto f_A(\mathrm{id}_A)).$$

In particular, considering the case $T = y(C)$ for some $C \in \mathcal{A}$, the map

$$y : y(C)(A) = \mathcal{A}(A,C) \to \mathrm{Nat}(y(A),y(C)) \qquad (t \mapsto y(t) \text{ for } t \in \mathcal{A}(A,C))$$

is an isomorphism. Hence, the functor $y : \mathcal{A} \to \mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \underline{\mathrm{Set}})$ is fully faithful.

(1.1.7) As the functor y is a full embedding, we sometimes identify $\mathcal{A}$ with the full subcategory $y(\mathcal{A})$ of $\mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \underline{\mathrm{Set}})$, and $y(A)$ is denoted simply by A for $A \in \mathcal{A}$. For $F \in \mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \underline{\mathrm{Set}})$, we have $A \in \mathcal{A}$ such that $F \cong y(A)$ is unique up to isomorphisms, if it exists. If such an A exists, then we say that F is *representable*, and we say that F is represented by A. Thus, $F \in \mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \underline{\mathrm{Set}})$ is representable if and only if $F \in y(\mathcal{A})$.

Similarly, a covariant functor $G \in \mathrm{Func}(\mathcal{A}, \underline{\mathrm{Set}})$ is said to be representable if G is isomorphic to $\mathcal{A}(A, ?)$ for some $A \in \mathrm{ob}(\mathcal{A})$.

1.2 Adjoint functors and limits

(1.2.1) Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and $F : \mathcal{A} \to \mathcal{B}$ a functor with the right adjoint $G : \mathcal{B} \to \mathcal{A}$. Namely, let us assume that $G : \mathcal{B} \to \mathcal{A}$ is a functor, and there exists some isomorphism

$$\Phi_{A,B} : \mathcal{B}(FA, B) \to \mathcal{A}(A, GB)$$

which is natural with respect to A and B. When we define

$$u_A := \Phi_{A,FA}(1_{FA}) : A \to (GF)A$$

for $A \in \mathcal{A}$, then u is a natural map from $\mathrm{Id}_{\mathcal{A}}$ to GF. We call u the *unit* of adjunction. Similarly,

$$\varepsilon_B := \Phi^{-1}_{GB,B}(1_{GB}) : (FG)B \to B$$

is natural with respect to B, and we call $\varepsilon : FG \to \mathrm{Id}_{\mathcal{B}}$ the *counit* of adjunction. They satisfy the relation

$$(1.2.2) \qquad (\varepsilon F) \circ (Fu) = 1_F, \qquad (G\varepsilon) \circ (uG) = 1_G.$$

Conversely, if two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ and natural transformations $u : \mathrm{Id}_{\mathcal{A}} \to GF$ and $\varepsilon : FG \to \mathrm{Id}_{\mathcal{B}}$ are given and the relation (1.2.2) is satisfied, then an equivalence Φ is defined by $\Phi_{A,B}(f) := G(f) \circ u_A$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $f \in \mathcal{B}(FA, B)$, and G is right adjoint to F. The natural maps u and ε determined by this adjunction agree with the original ones, respectively.

Note that a functor right adjoint to F is unique up to equivalence. This follows easily from Lemma 1.1.6.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. It is easy to see that $\mathrm{Func}(\mathcal{A}, \mathcal{B})$ is a category with small hom sets, if $\mathcal{A}$ is svelte. An object of $\mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \mathcal{B})$ is sometimes referred as a ($\mathcal{B}$-valued) *presheaf* over $\mathcal{A}$.

Let $B \in \mathcal{B}$. The functor $c(B) \in \mathrm{Func}(\mathcal{A}, \mathcal{B})$ defined by $c(B)(A) = B$ and $c(B)(f) = \mathrm{id}_B$ for any $A \in \mathrm{ob}(\mathcal{A})$ and any $f \in \mathrm{Mor}(\mathcal{A})$ is called the *constant functor* with the constant value B. Thus, we obtain a functor

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$c(?) : \mathcal{B} \to \mathrm{Func}(\mathcal{A}, \mathcal{B})$. Let $F \in \mathrm{Func}(\mathcal{A}, \mathcal{B})$. If $\mathrm{Nat}(F, c(?)) : \mathcal{B} \to \underline{\mathrm{Set}}$ is representable, then the object in $\mathcal{B}$ which represents $\mathrm{Nat}(F, c(?))$ is called the *inductive limit* of F, and is denoted by $\varinjlim F$. We say that the category $\mathcal{B}$ has inductive limits if for any svelte category $\mathcal{A}$ and $F \in \mathrm{Func}(\mathcal{A}, \mathcal{B})$, the functor $\mathrm{Nat}(F, c(?))$ is representable. If $\mathcal{B}$ has inductive limits, then we can make $\varinjlim : \mathrm{Func}(\mathcal{A}, \mathcal{B}) \to \mathcal{B}$ be a functor so that $\varinjlim$ is a left adjoint functor of c via the isomorphism $\mathrm{Nat}(F, c(?)) \cong \mathcal{B}(\varinjlim F, ?)$.

Dually, we define $\varprojlim F$, the *projective limit* of F, to be the object of $\mathcal{B}$ which represents $\mathrm{Nat}(c(?), F)$. We say that $\mathcal{B}$ has projective limits if for any svelte category $\mathcal{A}$ and $F \in \mathrm{Func}(\mathcal{A}, \mathcal{B})$, the functor $\mathrm{Nat}(c(?), F)$ is representable. If $\mathcal{B}$ has projective limits, then $\varprojlim : \mathrm{Func}(\mathcal{A}, \mathcal{B}) \to \mathcal{B}$ is a functor which is right adjoint to c.

Note that $\underline{\mathrm{Set}}$ and $\underline{\mathrm{Ab}}$ have both inductive limits and projective limits.

Definition 1.2.3 Let I be a category. We say that I is *filtered* if

1 For any $i, j \in I$, there exists some $k \in I$ such that $I(i, k) \neq \emptyset \neq I(j, k)$.

2 For any $i, j \in I$ and $f, g \in I(i, j)$, there exists some $k \in I$ and some $h \in I(j, k)$ such that $hf = hg$.

(1.2.4) An element of $\mathrm{Func}(I, \mathcal{B})$ is said to be a *filtered inductive system* (resp. *filtered projective system*) if I (resp. I^{op}) is filtered.

We say that $\varinjlim F$ is a *filtered inductive limit* if F is a filtered inductive system (and if it exists).

Let J be a full subcategory of a filtered category I. We say that J is a *final subcategory* of I if for any $i \in I$, there exists some $j \in J$ such that $I(i, j) \neq \emptyset$. We also say that I is *cofinal* with J. If this is the case, the restriction $\mathrm{Nat}_I(F, c(?)) \to \mathrm{Nat}_J(F|_J, c(?))$ is an isomorphism for $F \in \mathrm{Func}(I, \mathcal{B})$. In particular, if $\varinjlim F|_J$ exists, then it agrees with $\varinjlim F$.

Let P be a preordered set. Then P is a small category by letting $P(a, b)$ be a singleton if $a \leq b$ and the empty set if $a \not\leq b$. An ordered set P is filtered as a small category if and only if P is a directed set.

(1.2.5) Let $M : I \to \mathcal{A}$ be a functor such that I is svelte and $\varinjlim M$ exists. Then there is a natural map $\eta_M : M \to c(\varinjlim M)$ corresponding to $\mathrm{id} : \varinjlim M \to \varinjlim M$ by the isomorphism $\mathcal{A}(\varinjlim M, \varinjlim M) \cong \mathrm{Nat}(M, c(\varinjlim M))$. We say that a functor $F : \mathcal{A} \to \mathcal{B}$ preserves inductive limits if for any $M : I \to \mathcal{A}$ such that I is svelte and $\varinjlim M$ exists, the inductive limit $\varinjlim (F \circ M)$ also exists, and the map $\varinjlim (F \circ M) \to F(\varinjlim M)$ which corresponds to the natural map

$$F \circ M \xrightarrow{F\eta_M} F \circ (c(\varinjlim M)) = c(F(\varinjlim M))$$

is an isomorphism.

Lemma 1.2.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, $F : \mathcal{A} \to \mathcal{B}$ a functor with the right adjoint G. Then the following hold:*

1 *F preserves inductive limits. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are abelian, then F is right exact (as biproducts and cokernels are inductive limits). Moreover, the isomorphism Φ is an isomorphism of abelian groups.*

1* *G preserves projective limits.*

2 *If $\mathcal{A}$ and $\mathcal{B}$ are abelian and F is right exact, then G preserves injective objects.*

3 *F is faithful if and only if u is a monomorphism (i.e., for any A, u_A is a monomorphism).*

4 *F is fully faithful if and only if u is an isomorphism. In particular, if F is fully faithful and $\mathcal{B}$ has projective limits, then $\mathcal{A}$ also has projective limits. In fact, we have $G \varprojlim Ff = \varprojlim f$.*

4* *If G is fully faithful and $\mathcal{A}$ has inductive limits, then $\mathcal{B}$ has inductive limits. In fact, $F \varinjlim Gf = \varinjlim f$.*

Proof. We prove only **3** and **4**. Note that u is a monomorphism (resp. an isomorphism) if and only if $u_* : \mathcal{A}(A, A') \to \mathcal{A}(A, GFA')$ given by $u_*(f) = u \circ f$ is injective (resp. bijective) for any $A, A' \in \mathcal{A}$. We have for $f \in \mathcal{A}(A, A')$ by the naturality of Φ and the naturality of u,

$$\begin{aligned}\Phi_{A,FA'}(F(f)) = \Phi_{A,FA'}(\mathcal{B}(FA, F(f))(1_{FA})) &= \mathcal{A}(A, (GF)(f))(\Phi_{A,FA}(1_{FA})) \\ &= \mathcal{A}(A, (GF)(f))(u_A) = (GF)(f) \circ u_A = u_{A'} \circ f = u_*(f).\end{aligned}$$

As Φ is an isomorphism, u_* is injective (resp. bijective) if and only if

$$F : \mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$$

has the same property. Hence, **3** and the first part of **4** follow.

If $u : \mathrm{Id}_{\mathcal{A}} \to GF$ is an isomorphism and $\mathcal{B}$ has projective limits, then for $f \in \mathrm{Func}(I, \mathcal{A})$, we have $G \varprojlim Ff \cong \varprojlim GFf \cong \varprojlim f$, and $\mathcal{A}$ also has projective limits. □

1.3 Exact categories

Let $\mathcal{A}$ be an additive category.

Definition 1.3.1 We say that $\mathcal{A}$ is an *exact category* if two classes of morphisms $\mathcal{E}_m$ and $\mathcal{E}_e$ of $\mathcal{A}$ are specified and the following conditions are satisfied:

E1 If $i, j \in \mathcal{E}_m$ and ij is defined, then $ij \in \mathcal{E}_m$.

E1* If $p, q \in \mathcal{E}_e$ and pq is defined, then $pq \in \mathcal{E}_e$.

E2 Any split monomorphism which has a cokernel belongs to $\mathcal{E}_m$.

E3 If $i \in \mathcal{E}_m$, then i has a cokernel, $\operatorname{coker} i \in \mathcal{E}_e$, and i is a kernel of $\operatorname{coker} i$.

E3* If $p \in \mathcal{E}_e$, then p has a kernel, $\ker p \in \mathcal{E}_m$, and p is a cokernel of $\ker p$.

E4 Let $p : B \to C$ and $g : C' \to C$ be morphisms of $\mathcal{A}$, and assume that $p \in \mathcal{E}_e$. Then there exists a pull-back of p and g, and the base change p' of p belongs to $\mathcal{E}_e$.

E4* Let $i : A \to B$ and $f : A \to A'$ be morphisms of $\mathcal{A}$, and assume that $i \in \mathcal{E}_m$. Then there exists a push-out of i and f, and the cobase change i' of i belongs to $\mathcal{E}_m$.

E5 If $i : A \to B$ and $f : B \to B'$ are morphisms of $\mathcal{A}$, $fi \in \mathcal{E}_m$, and i has a cokernel, then $i \in \mathcal{E}_m$.

E5* If $p : B \to C$ and $g : B' \to B$ are morphisms of $\mathcal{A}$, $pg \in \mathcal{E}_e$, and p has a kernel, then $p \in \mathcal{E}_e$.

(1.3.2) A sequence of morphisms

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{1.3.3}$$

of an exact category $\mathcal{A}$ is called a *short exact sequence* if i is a kernel of p, p is a cokernel of i, and $i \in \mathcal{E}_m$ (or equivalently, $p \in \mathcal{E}_e$). We also say that (i, p) is a short exact sequence. Let us denote the set (not necessarily small) of short exact sequences in $\mathcal{A}$ by $\mathcal{E}$. Then $\mathcal{E}_m$ is the set of morphisms of $\mathcal{A}$ such that there exists some morphism p with $(i, p) \in \mathcal{E}$. Similarly, $\mathcal{E}_e$ is also determined by $\mathcal{E}$. Hence, we also say that $(\mathcal{A}, \mathcal{E})$ is an exact category. Moreover, $\mathcal{E}_e$ is the set of morphisms which are cokernels of some morphisms of $\mathcal{E}_m$. Similarly, $\mathcal{E}_e$ is determined by $\mathcal{E}_m$. Hence, any one of $\mathcal{E}_e$, $\mathcal{E}_m$ and $\mathcal{E}$ determines the others.

(1.3.4) We say that a morphism f of an exact category $\mathcal{A}$ is *admissible* if there is a factorization $f = ip$ such that $i \in \mathcal{E}_m$ and $p \in \mathcal{E}_e$. For any epi-mono decomposition $f = i'p'$ of an admissible morphism $f = ip$, there exists some isomorphism α such that $i' = i\alpha$ and $p' = \alpha^{-1}p$. An admissible mono(resp. epi-)morphism is nothing but a morphism of $\mathcal{E}_m$ (resp. $\mathcal{E}_e$).

We say that a sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$ is exact if f and g are admissible with epi-mono decompositions $f = ip$ and $g = jq$, respectively, such that $(i, q) \in \mathcal{E}$. We also say that $[f, g]$ is exact. A complex

$$\cdots \to A^{i-1} \xrightarrow{\partial^{i-1}} A^i \xrightarrow{\partial^i} A^{i+1} \to \cdots$$

in $\mathcal{A}$ is called exact if $[\partial^{i-1}, \partial^i]$ is exact for any $i \in \mathbb{Z}$.

(1.3.5) An additive functor between exact categories is called an *exact functor* if it preserves short exact sequences. An additive functor between exact categories F is called *half exact* if $[Fi, Fp]$ is exact for any short exact sequence (i, p). Similarly, left and right exact functors are also defined.

Definition 1.3.6 An additive category $\mathcal{A}$ is called *semisaturated* (resp. *Karoubian* or sometimes *saturated*) if any split epimorphism has a kernel (resp. any projector (i.e., idempotent endomorphism) has an image).

(1.3.7) An abelian category is Karoubian. Any Karoubian additive category is semisaturated. If $\mathcal{A}$ is semisaturated (resp. Karoubian), then so is $\mathcal{A}^{\mathrm{op}}$.

Definition 1.3.8 Let $\mathcal{A}$ be an exact category, and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. We say that $\mathcal{B}$ is closed under extensions (resp. monocokernels, epikernels) in $\mathcal{A}$ if $\mathcal{B}$ contains some null object of $\mathcal{A}$, and $A, C \in \mathcal{B}$ (resp. $A, B \in \mathcal{B}$, $B, C \in \mathcal{B}$) implies $A, B, C \in \mathcal{B}$ for any exact sequence (1.3.3) in $\mathcal{A}$. If $B \in \mathcal{B}$ implies $A \in \mathcal{B}$ (resp. $C \in \mathcal{B}$) and $\mathcal{B}$ is non-empty, then we say that $\mathcal{B}$ is closed under subobjects (resp. quotients). We say that $\mathcal{B}$ is closed under subquotients if $\mathcal{B}$ is closed under both subobjects and quotient objects. If $A \in \mathcal{B}$ (resp. $D \in \mathcal{B}$) for any exact sequence

$$0 \to A \to B \to C \to D \to 0$$

in $\mathcal{A}$ with $B, C \in \mathcal{B}$, then we say that $\mathcal{B}$ is closed under kernels (resp. cokernels). If $\mathcal{B}$ is closed under kernels, cokernels and extensions, then we say that $\mathcal{B}$ is a *thick subcategory* of $\mathcal{A}$. A thick subcategory closed under subquotients is said to be *very thick*.

(1.3.9) If $\mathcal{B}$ is closed under extensions, then it is closed under isomorphisms and finite direct sums, and $\mathcal{B}$ itself is an additive category so that the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is additive. A thick subcategory of an abelian category is abelian, and the inclusion functor is fully faithful and exact.

Exercise 1.3.10 Prove the following.

1 Let $\mathcal{A}$ be an abelian category. Letting $\mathcal{E}_m$ be its set of monomorphisms and $\mathcal{E}_e$ its set of epimorphisms, $\mathcal{A}$ is an exact category. In this case, $\mathcal{E}$ is the set of (usual) short exact sequences.

2 If $(\mathcal{A}, \mathcal{E})$ is an exact category, then $(\mathcal{A}^{\text{op}}, \mathcal{E}^{\text{op}})$ is also an exact category with $\mathcal{E}_m^{\text{op}}$ and $\mathcal{E}_e^{\text{op}}$ its set of admissible epimorphisms and monomorphisms, respectively.

3 Let $\mathcal{B}'$ be a full subcategory of an exact category $\mathcal{B}$ closed under extensions. Defining a short exact sequence in $\mathcal{B}'$ to be a short exact sequence in $\mathcal{B}$ consisting of objects of $\mathcal{B}'$, $\mathcal{B}'$ is an exact category, and the inclusion $\mathcal{B}' \hookrightarrow \mathcal{B}$ is exact.

3′ Let $F : \mathcal{B} \to \mathcal{C}$ be a half exact functor between exact categories. Then $\operatorname{Ker} F := \{B \in \mathcal{B} \mid FB \cong 0\}$ is closed under extensions in $\mathcal{B}$.

4 If $(\mathcal{A}, \mathcal{E}_\lambda)$ is a family of exact categories, then $(\mathcal{A}, \bigcap_\lambda \mathcal{E}_\lambda)$ is an exact category.

4′ Let $F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{B}, \mathcal{E}')$ be an exact functor between exact categories. If $\mathcal{E}'' \subset \mathcal{E}'$ and $(\mathcal{B}, \mathcal{E}'')$ is also an exact category, then defining

$$\mathcal{E}''' := \{E \in \mathcal{E} \mid F(E) \in \mathcal{E}''\},$$

$(\mathcal{A}, \mathcal{E}''')$ is also an exact category. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are abelian categories (with the structures of exact categories as in **1**) and when we denote the set of exact sequences E in $\mathcal{A}$ such that $F(E)$ is split exact by $\mathcal{E}(F)$, then $(\mathcal{A}, \mathcal{E}(F))$ is an exact category.

Any exact category $\mathcal{B}'$ is always produced as in **3** above, with $\mathcal{B}$ abelian (but not necessarily a $\mathcal{U}$-category), as follows.

Theorem 1.3.11 *Let $\mathcal{A}$ be an exact R-category. Then the category $\mathcal{B} :=$ $\operatorname{Sex}_R(\mathcal{A}^{\text{op}}, {}_R\mathbb{M})$ of contravariant left exact R-functors from $\mathcal{A}$ to the category of R-modules ${}_R\mathbb{M}$ is abelian (the notation* Sex *is due to Gabriel, and explained thus:* sinister exact*). The Yoneda embedding $y : \mathcal{A} \to \mathcal{B}$ is an R-equivalence from $\mathcal{A}$ to a full subcategory $\mathcal{B}'$ of $\mathcal{B}$ closed under extensions. Moreover, for a sequence of morphisms (i, p) in $\mathcal{B}$, (i, p) is a short exact sequence if and only if (yi, yp) is a short exact sequence. If moreover $\mathcal{A}$ is semisaturated, then $\mathcal{B}'$ is closed under epikernels in $\mathcal{B}$.*

This theorem was proved by Quillen [126]. For the proof, see [141]. The Yoneda embedding y is also called the *Gabriel–Quillen embedding* in this case.

Corollary 1.3.12 *The five lemma and the 3×3 lemma are true in exact categories.*

Corollary 1.3.13 *The canonical functor from the category of Karoubian (resp. semisaturated) exact categories to the category of exact categories has* *a left adjoint. Namely, if $\mathcal{B}$ is an exact category, then there exists a (unique) Karoubian exact category $\mathcal{B}^s$ (resp. semisaturated exact category $\mathcal{B}^{ss}$) and an exact functor $f : \mathcal{B} \to \mathcal{B}^s$ (resp. $f' : \mathcal{B} \to \mathcal{B}^{ss}$) such that for any Karoubian (resp. semisaturated) exact category $\mathcal{A}$ and any exact functor $g : \mathcal{B} \to \mathcal{A}$, there exists a unique exact functor $h : \mathcal{B}^s \to \mathcal{A}$ (resp. $h' : \mathcal{B}^{ss} \to \mathcal{A}$) such that hf (resp. $h'f'$) and g are equivalent.*

For an exact category $\mathcal{B}$, we call $\mathcal{B}^s$ (resp. $\mathcal{B}^{ss}$) the *saturation* (resp. *semisaturation*) of $\mathcal{B}$.

1.4 Derived categories and derived functors

(1.4.1) Let $\mathcal{A}$ be an additive category. We only treat cohomological chain complexes here. We say that $\mathbb{F} = (F^i, d^i)$ is a chain complex in $\mathcal{A}$ if $(F^i)_{i\in\mathbb{Z}}$ is a collection of objects of $\mathcal{A}$, $d^i \in \mathcal{A}(F^i, F^{i+1})$, and $d^{i+1} \circ d^i = 0$ $(i \in \mathbb{Z})$. A homological complex can be viewed as a cohomological complex with the identification $F^i := F_{-i}$.

A chain complex $\mathbb{F}$ is said to be *bounded below* (resp. *bounded above*) if $F^i = 0$ for $i \ll 0$ (resp. $i \gg 0$). It is said to be *bounded* if it is both bounded below and bounded above. The length of a complex $\mathbb{F}$ is defined to be $\sup\{i - j \mid F_i \neq 0,\ F_j \neq 0\}$, so the length of $\mathbb{F}$ is not ∞ if and only if $\mathbb{F}$ is bounded. As a convention, we define the length of the zero complex to be $-\infty$. The category $C(\mathcal{A})$ of chain complexes and chain maps in $\mathcal{A}$ is an additive category. If $\mathcal{A}$ is an R-category, then so is $C(\mathcal{A})$. The full subcategory of complexes bounded below (resp. bounded above, bounded) is denoted by $C^+(\mathcal{A})$ (resp. $C^-(\mathcal{A})$, $C^b(\mathcal{A})$).

(1.4.2) Let $\mathbb{F}$ and $\mathbb{G}$ be complexes in $\mathcal{A}$. Then we have the following chain complexes obtained from $\mathbb{F}$ and $\mathbb{G}$.

$\mathbb{F}[n]$: Defining $F[n]^i := F^{n+i}$ and $d^i_{\mathbb{F}[n]} := (-1)^n d^{n+i}_{\mathbb{F}}$, we have a complex $\mathbb{F}[n]$ in $\mathcal{A}$.

$\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F}, \mathbb{G})$: We define $\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F}, \mathbb{G})^i := \prod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(F^n, G^{n+i})$, and

$$d^i_{\operatorname{Hom}}((f^n)_n) := (d^{n+i}_{\mathbb{G}} f^n - (-1)^i f^{n+1} d^n_{\mathbb{F}})_n.$$

Then $\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F}, \mathbb{G})$ is a complex of abelian groups. If $\mathcal{A}$ is an R-category, then it is a complex of R-modules.

$\mathbb{F} \otimes^\bullet_R \mathbb{G}$: Let $\mathcal{A}$ be the category of R-modules ${}_R\mathbb{M}$. Then defining $(\mathbb{F} \otimes^\bullet_R \mathbb{G})^i := \bigoplus_{m+n=i} F^m \otimes G^n$ and $d^i(f^m \otimes g^n) = d^m f^m \otimes g^n + (-1)^m f^m \otimes d^n g^n$, we get a complex $\mathbb{F} \otimes^\bullet_R \mathbb{G}$ in $\mathcal{A}$. A natural map $H^m(\mathbb{F}) \otimes H^n(\mathbb{G}) \to H^{m+n}(\mathbb{F} \otimes^\bullet_R \mathbb{G})$ is induced.

The following is checked easily.

Exercise 1.4.3 The composition

$$\text{(1.4.4)} \qquad \operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G},\mathbb{H}) \otimes^\bullet_{\mathbb{Z}} \operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F},\mathbb{G}) \to \operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F},\mathbb{H})$$

is a chain map.

Note that an n-cocycle in $\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F},\mathbb{G})$ is nothing but a chain map from $\mathbb{F}$ to $\mathbb{G}[n]$. An n-coboundary is a null homotopic chain map from $\mathbb{F}$ to $\mathbb{G}[n]$.

Letting chain complexes in $\mathcal{A}$ be the objects and $H^0(\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{F},\mathbb{G}))$ the hom set from $\mathbb{F}$ to $\mathbb{G}$, we get a category $K(\mathcal{A})$. The composition is that of chain maps. It is well-defined, because it agrees with the map of cohomology induced by (1.4.4). The full subcategory of $K(\mathcal{A})$ consisting of complexes bounded below (resp. bounded above, bounded) is denoted by $K^+(\mathcal{A})$ (resp. $K^-(\mathcal{A})$, $K^b(\mathcal{A})$).

(1.4.5) If $\mathcal{A}$ is an exact category, then $K(\mathcal{A})$ has the structure of a triangulated category [120]. The translation functor T is given by $T(\mathbb{F}) := \mathbb{F}[1]$. For basics on triangulated categories, see [143].

Let ? be either b, $+$, $-$, or $\emptyset$. We denote the full subcategory of $K^?(\mathcal{A})$ consisting of all exact sequences in $K^?(\mathcal{A})$ by $E^?(\mathcal{A})$.

Proposition 1.4.6 (Neeman) *With the notation above, $E^?(\mathcal{A})$ is a triangulated subcategory of $K^?(\mathcal{A})$. If $\mathcal{A}$ is Karoubian, then $E^?(\mathcal{A})$ is épaisse. If $? = +, -, b$ and $\mathcal{A}$ is semisaturated, then $E^?(\mathcal{A})$ is épaisse.*

(1.4.7) The quotient $K^?(\mathcal{A})/E^?(\mathcal{A})^e$ (for definition, see [143]) is denoted by $D^?(\mathcal{A})$, and it is called the *derived category* of $\mathcal{A}$, where $E^?(\mathcal{A})^e$ denotes the épaisse closure of $E^?(\mathcal{A})$. By Rickard's criterion [130], $E^?(\mathcal{A})^e$ is the set of direct summands of exact sequences in $K^?(\mathcal{A})$. Note that $D^?(\mathcal{A})$ may not be a category with small hom sets any more. As an easy criterion, note that $D^?(\mathcal{A})$ is a category with small hom sets if $\mathcal{A}$ is svelte (follows from [143, p. 298]).

Let $f : \mathbb{F} \to \mathbb{G}$ be a morphism of $C(\mathcal{A})$. We say that f is a *quasi-isomorphism* if the mapping cone $C(f)$ is a direct summand of an exact sequence. If $f - f'$ is null homotopic, then $C(f)$ and $C(f')$ are isomorphic in $C(\mathcal{A})$. So the notion of quasi-isomorphism is also defined for morphisms of $K(\mathcal{A})$. Note that $D^?(\mathcal{A})$ is obtained by localizing quasi-isomorphisms of $K^?(\mathcal{A})$.

Definition 1.4.8 Let $\mathcal{A}$ be an exact category, and $A \in \operatorname{ob}(\mathcal{A})$. We say that an object I of $\mathcal{A}$ is injective if any admissible monomorphism $I \to B$ splits. We say that $\mathcal{A}$ has enough injectives if for any $A \in \operatorname{ob}(\mathcal{A})$, there exists some injective object I of $\mathcal{A}$ and an admissible monomorphism $A \to I$.