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Wavelets

Calderón–Zygmund and multilinear operators

Yves Meyer

Professor, Ceremade, Université Paris–Dauphine

Ronald Coifman

Yale University

Translated by David Salinger

University of Leeds

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"Ce à quoi l'un s'était failli, l'autre est arrivé et ce qui était inconnu à un siècle, le siècle suivant l'a éclairci, et les sciences et les arts ne se jettent pas en moule mais se forment et figurent en les maniant et polissant à plusieurs fois [...] Ce que ma force ne peut découvrir, je ne laisse pas de le sonder et essayer et, en retastant et pétrissant cette nouvelle matière, la remuant et l'eschaufant, j'ouvre à celui qui me suit quelque facilité et la lui rends plus souple et plus maniable. Autant en fera le second au tiers qui est cause que la difficulté ne me doit pas désespérer, ni aussi peu mon impuissance ..."

Montaigne, *Les Essais*, Livre II, Chapitre XII.

"Where someone failed, another has succeeded; what was unknown in one century, the next has discovered; science and the arts do not grind themselves into uniformity, but gain shape and regularity by carving and polishing repeatedly [...] What my own strength has not been able to uncover, I cease not from working at and trying out and, by reshaping and solidifying this new material, in moulding and heating it, I bequeath to him who follows some facility and make it the more supple and malleable for him. The second will do the same for the third, which is why difficulty does not make me despair, nor, any the more, my own weakness..."

Translator's note

This book is a translation of *Ondelettes et Opérateurs*, Volume II, *Opérateurs de Calderón-Zygmund*, by Yves Meyer, and Volume III, *Opérateurs multilinéaires*, by R.R. Coifman and Yves Meyer. The original numbering of the chapters and of the theorems has been retained, so that it is still possible to follow the forward references in Volume I. Chapter 12 is where Volume III of the French version started. The references to *Wavelets and Operators* (Cambridge University Press, 1992), are to the translation of Volume I, *Ondelettes*, by Yves Meyer.

David Salinger, Leeds, June 1996.

Preface to the English Edition

There has been great progress during the few years which separate the first edition of this work from the translation.

1. In the area of **multilinear operators**, Pierre Louis Lions made the following conjecture. We consider two (arbitrary) vector fields $E(x) = (E_1(x), \ldots, E_n(x))$ and $B(x) = (B_1(x), \ldots, B_n(x))$, satisfying the following conditions:
$$E_j(x) \in L^p(\mathbb{R}^n) \qquad \text{and} \qquad B_j(x) \in L^q(\mathbb{R}^n)\,,$$
where $1 \leq j \leq n$, $1 < p < \infty$, $1/p + 1/q = 1$, and
$$\operatorname{div} E(x) = 0 \qquad \text{and} \qquad \operatorname{curl} B(x) = 0\,,$$
in the sense of distributions.
Then the scalar product $E \cdot B(x) = E_1(x)B_1(x) + \cdots + E_n(x)B_n(x)$, of the vector fields must belong to the Hardy space $\mathbb{H}^1(\mathbb{R}^n)$.
The reader may refer to [E8], where there are several proofs of this result and to [E11], where an application to partial differential equations is to be found.

2. For the **Cauchy Integral** on Lipschitz curves, Mark S. Melnikov and Joan Verdera have found a new, extraordinary proof. This proof is astonishing because it makes no use of the BMO space or of Carleson measures. Instead, an essential rôle is played by a very simple and surprising lemma about the geometry of the complex plane [E18].

3. For more about **potential theory** on Lipschitz domains and Verchota's thesis, the reader may turn to C. Kenig's excellent recent work [E15].

4. Can wavelets play a part in the study or the understanding of the **Navier-Stokes equations**? We still do not know the answer to this question. P. Federbush [E13] has undertaken the ambitious programme of solving the Navier-Stokes equations by a method of Faedo-Galerkin, using wavelet bases in which the wavelets have zero divergence. These bases were constructed by G. Battle and P. Federbush and, independently, by P.G. Lemarié-Rieusset [E17].

   Federbush's approach is analysed and discussed in M. Cannone's elegant text [E5].

5. Paradifferential operators are flourishing as a tool in the study and solution of non-linear partial differential equations. The reader may consult [E6] and [E21].

6. P. Auscher and Ph. Tchamitchian continue the study of problems raised by Kato's conjecture. These authors have obtained remarkable results in this direction without, however, entirely solving the original problem [E2].

7. In the even more applied areas of **Numerical Analysis** and **Statistics**, essential results are rooted in the analysis of Calderón-Zygmund operators in appropriate wavelet bases [E3], [E10].

Yves Meyer, Paris, May 1996.

Introduction

Confronted with an orthonormal basis e_j, $j \in J$, of the Hilbert space $H = L^2(\mathbb{R}^n)$, it is impossible to resist the temptation to study those operators $T : H \to H$ which are diagonal, or almost diagonal, with respect to that particular basis. We can then hope that those operators are familiar from other contexts, which would give the whole situation that coherence beloved of scientists. Unfortunately, until now, the attempt has been in vain. Diagonal operators corresponding to the usual orthonormal bases are generically pathological, and therefore not of interest. For example, in the trigonometric system, the diagonal operators are those satisfying $T(e^{ikx}) = m_k e^{ikx}$, where m_k is a bounded sequence; they are pathological unless, for example, Marcinkiewicz's condition $|m_{k+1} - m_k| \leq C/|k|$ is satisfied. In this case, T is related to a pseudo-differential operator and constitutes a first example of what we shall call **Calderón-Zygmund operators**.

Orthonormal wavelet bases provide the first and—as far as we know—the unique example of an orthonormal basis with interesting diagonal, or almost-diagonal, operators. Those operators are already known, in another context, as Calderón-Zygmund operators. This remarkable fact explains the success of wavelet series compared to other orthonormal series. A change of wavelet coefficients $a_k \to m_k a_k$, where m_k is a bounded sequence, does not have uncontrollable consequences for the sum $f(x) = \sum a_k w_k(x)$ of the wavelet series. The sum $f(x)$ is transformed into $g(x) = T[f](x)$ and the remarkable properties of the Calderón-Zygmund operators let us give a precise description of what happens to $f(x)$. Wavelet decompositions are thus robust, and their robustness comes from the relationship between wavelets and operators which we have just described.

But Calderón-Zygmund operators had been studied well before orthonormal wavelet bases came to light: they are the subject of an independent theory which we shall expound completely and autonomously in Chapters 7 to 11. So the reader may attempt this volume straight away, retaining from *Wavelets and Operators* just the existence of orthonormal wavelet bases. Calderón-Zygmund operators have a special relationship with wavelets and with classical pseudo-differential operators, of which they are a remarkable generalization. In fact, the Calderón-Zygmund operators are found "beyond the pseudo-differential operators".

Multilinear analysis is one of the routes into the non-linear problems studied in Chapters 12 to 16. This route is only possible for those non-linear problems with a holomorphic structure, enabling them to be decomposed into a series of multilinear terms of increasing complexity. This approach needs some care, because the holomorphic structure can only be established after the event, that is, once we have shown that the series of multilinear terms converges. A. Calderón was the pioneer of this method of attack, and some of the examples we give here are part of his programme. Others are due to T. Kato: the discovery, by A. McIntosh, of the relationship between Calderón's programme and Kato's has been a source of progress in recent years.

The multilinear operators encountered in the problems referred to above turn out to be Calderón-Zygmund operators, whose continuity is established using the earlier chapters of this volume. Wavelets make a final appearance—as eigenfunctions of certain realizations of paraproducts—in our last chapter, which is devoted to J.M. Bony's theory of paradifferential operators.

For the greater convenience of the reader, the Introduction to *Wavelets and Operators*, which also serves as a general introduction to the present volume, is reproduced in the next few pages.

Introduction to *Wavelets and Operators*

For many years, the sine, cosine and imaginary exponential functions have been the basic functions of analysis. The sequence $(2\pi)^{-1/2}e^{ikx}$, $k = 0, \pm1, \pm2, \ldots$ forms an orthonormal basis of the standard space $L^2[0, 2\pi]$; Fourier series are the linear combinations $\sum a_k e^{ikx}$. Their study has been, and remains, an unquenchable source of problems and discoveries in mathematical analysis. The problems arise from the absence of a good dictionary for translating the properties of a function into those of its Fourier coefficients. Here is an example of the kind of difficulty that occurs. J.P. Kahane, Y. Katznelson and K. de Leeuw have shown ([150]) that, to get a continuous function $g(x)$ from an arbitrary square-summable function $f(x)$, it is sufficient to increase—or leave unchanged—the moduli of the Fourier coefficients of $f(x)$ and to adjust their phases judiciously. It is thus impossible to predict the properties (size, regularity) of a function solely from knowledge of the order of magnitude of its Fourier coefficients. Indeed it is still difficult if we know the Fourier coefficients explicitly, and many problems are still open.

At the beginning of the 1980s, many scientists were already using "wavelets" as an alternative to traditional Fourier analysis. This alternative gave grounds for hoping for simpler numerical analysis and more robust synthesis of certain transitory phenomena. The "wavelets" of J.S. Liénard or of X. Rodet ([167], [206]) were used for numerical treatment of acoustic signals (words or music) and those of J. Morlet ([124]) for stocking and interpreting seismic signals gathered in the course of oil prospecting expeditions. Among mathematicians, research was just as active: to mention only the most striking, R.R. Coifman and G. Weiss ([75]) invented the "atoms" and "molecules" which were to form the basic building blocks of various function spaces, the rules of assembly being clearly defined and easy to use. Certain of these atomic decompositions could, moreover, be obtained by making a discrete version of a well-known identity, due to A. Calderón, in which "wavelets" were implicitly involved. That identity was later rediscovered by Morlet and his collaborators .... Lastly, L. Carleson used functions very similar to "wavelets" in order to construct an unconditional basis of the H^1 space of E.M. Stein and G. Weiss.

These separate investigations had such a "family resemblance" that it seemed necessary to gather them together into a coherent theory, mathematically well-founded and, at the same time, universally applicable. The **orthonormal wavelet bases**, whose construction is given in the present volume, are a replacement for the empirical "wavelets" of Liénard, Morlet and Rodet.

The same orthonormal wavelet bases give direct access to the "atomic decompositions" of Coifman and Weiss, which are thus—for the first time—related to constructions of **unconditional bases** of the standard spaces of functions and distributions. The wavelet bases are universally applicable: "everything that comes to hand", whether function or distribution, is the sum of a wavelet series and, contrary to what happens with Fourier series, the coefficients of the wavelet series translate the properties of the function or distribution simply, precisely and faithfully.

So we have a new tool at our command, an instrument that lets us perform, without thinking, the delicate constructions that could not formerly be achieved without recourse to lacunary, or random, Fourier series. The exceptional properties of the sums of these special series become the everyday properties of generic sums of wavelet series.

The algorithms for analysis by, and synthesis of, orthogonal wavelet series will, doubtless, play an important rôle in many different branches of science and technology. Mathematicians, physicists, and engineers who want to know everything about wavelets now have the present volume (*Wavelets and Operators*) of this work at their disposal.

Wavelets: Calderón-Zygmund and Multilinear Operators is addressed more specifically to an audience of mathematicians. It deals with the operators associated with wavelets. G. Weiss has shown that the study of the operators acting on a space of functions or distributions can become very simple when the elements of the space admit "atomic decompositions". He writes "many problems in analysis have natural formulations as questions of continuity of linear operators defined on spaces of functions or distributions. Such questions can often be answered by rather straightforward techniques if they can first be reduced to the study of the operator on an appropriate class of simple functions which, in some convenient sense, generate the entire space." When these "simple elements" were the functions e^{ikx} of the trigonometric system, the bounded operators T on L^2, which were diagonal with respect to the trigonometric system, did not have any other interesting property (with the exception of translation-invariance, which follows immediately from the definition). It was then necessary to impose quite precise conditions on the eigenvalues of T in order to extend such an operator to other function spaces: the first results in this direction were obtained by J. Marcinkiewicz.

However, the bounded operators which can be diagonalized exactly or approximately, with respect to the wavelet basis, form an algebra **A** of bounded operators on L^2 and the well-known **Calderón-Zygmund real-variable methods** enable the operators of **A** to be extended to other spaces of functions or distributions. The algebra **A**, which extends the pseudo-differential operators in a natural sense, is strictly contained in the set **C** of operators whose study has been recommended by Calderón. Work on these operators should enable us to solve several outstanding classical problems in complex analysis and partial differential equations.

Here is a slightly more precise description of the set **C**, the delicate construction of which we have called "Calderón's programme". After having invented, together with A. Zygmund, what was to become the classical pseudo-differential calculus, Calderón intended to extend the field of application systematically, by weakening, as far as possible, the regularity hypotheses necessary for the algorithms to work.

The fundamental—and unexpected—discovery made by Calderón was the existence of a limit to the search for minimal hypotheses of regularity. There is a "natural boundary" which cannot be transgressed, and the extension of operators to this boundary is precisely the analytic extension of holomorphic functions on certain Banach spaces, as we shall show in Chapter 13.

Chapters 7–9 are devoted to the study and then the construction of the set **C** of operators of Calderón's programme. We call them the **Calderón-Zygmund operators**, although they are very different from the "historical operators" studied by Calderón and Zygmund in the 1950s and 60s.

Just like these "historical operators", those we consider can be defined by singular integrals, in a new sense which we clarify in Chapter 7. To go beyond the context of convolution operators, it becomes indispensable to have a criterion for L^2 continuity, without which the theory collapses like a house of cards. One such criterion is the well-known $T(1)$ theorem of G. David and J.L. Journé, which we shall prove in Chapter 8. The $T(1)$ Theorem replaces the Fourier transform, whose use remains restricted to convolution operators.

Unfortunately, the $T(1)$ theorem, although giving a necessary and sufficient condition, is not directly applicable to the most interesting operators of the set **C** of Calderón's programme. We do not know why that is. The operators in question have, however, a very special non-linear structure, which, when correctly exploited, allows us to pass from the "local" results given by David and Journé's theorem to the "global" theorems necessary for the functioning of Calderón's programme.

In Chapters 12–16, and also in Chapter 9, we have given the most beautiful of the applications of Calderón's programme. First comes the celebrated pseudo-differential calculus, initiated by Calderón, which, at present, has interesting and important applications to non-linear partial differential equations.

Then we pass to complex analysis and the Hardy spaces associated with Lipschitz domains of the complex plane. The object of Chapter 12 is the study of the Cauchy operator on rectifiable curves. We then examine the problem, posed by T. Kato, of determining the domain of the square roots of second order pseudo-differential operators, in the accretive case.

After that, we give an account of the results of B. Dahlberg, D. Jerison, C. Kenig and G. Verchota relative to the Dirichlet and Neumann problems in Lipschitz open sets.

We end with a brief presentation of J.M. Bony's paradifferential operators, which serve to analyse the regularity of non-linear partial differential operators.

Wavelets reappear, in a surprising way, as the eigenfunctions of certain paradifferential operators. Correctly handled, they remain present in the study of Hardy spaces and the Cauchy operator on a Lipschitz curve: the operator is "almost diagonal" with respect to a wavelet basis specially designed for complex analysis on that curve (Chapter 11). The construction of wavelet bases is thus sufficiently supple to be adaptable to differing geometric situations: we also obtain "non-orthogonal wavelet bases". At present, there is no universal basis which can simultaneously be used in the analysis of all the operators of Calderón's class **C**.

J.O. Strömberg was the first to construct an orthonormal basis of $L^2(\mathbb{R})$, of the form $2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, where, for each $m \in \mathbb{N}$, the function $\psi(x)$ was of class C^m and decreased exponentially at infinity.

Subsequent work on orthogonal wavelets has not followed the path discovered by Strömberg. That work is, essentially, due to I. Daubechies, P.G. Lemarié, S. Mallat, and the author. It is given here, with care and with complete proofs.

As far as operators are concerned, the well-known results of Calderón, Zygmund, and Cotlar will be described in Chapter 7, in the new context of the set **C**.

The other names the reader of this work will often encounter are J.M. Bony, G. David, P. Jones, J.L. Journé, C. Kenig, T. Murai, and S. Semmes.

The division into two volumes will allow this work to be read in several ways. As we have already suggested, the reader may wish to go no further than *Wavelets and Operators*, which surveys our present knowledge about wavelets. But the first part of *Wavelets: Calderón-Zygmund and Multilinear Operators* may also be read directly, assuming only the results quoted in the introductions of the first six chapters. Finally, the reader can go straight to Chapters 12, 13, 14, 15, or 16, because each of them forms a coherent account of a subject of independent interest (complex analysis, holomorphic functionals on Banach spaces, Kato theory, elliptic partial differential equations in Lipschitz domains, and, lastly, non-linear partial differential equations). The thread linking these different themes is, quite clearly, the use of wavelets in Calderón's programme in operator theory.

These books have been written at a level appropriate for first-year postgraduates, and we have tested them in France, and in the U.S.A., on various audiences of mathematicians and engineers. To read these books, it is therefore not necessary to have studied the remarkable treatises by E. Stein and G. Weiss ([221)], E. Stein ([217]), or J. Garcia-Cuerva and J.L. Rubio de Francia ([115]), not to mention the fundamental text and reference by Zygmund ([239]).

R. Coifman helped me to recognize the importance of Calderón's programme. Since the summer of 1974, our scientific collaboration has been devoted to its realization, and these books have been one of our projects. If our own work no longer appears in its original form here, it is because our zeal and enthusiasm have communicated themselves to younger research workers, who have found more elegant solutions to the problems we had been determined to resolve.