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Global Attractors in Abstract Parabolic Problems

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK http://www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011–4211, USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

First published 2000

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this book is available from the British Library

ISBN 0 521 79424 2 paperback

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Preface

The past forty years have witnessed an intensive study of problems in mathematical physics governed by dissipative equations and much progress has been achieved. Two already existing branches of mathematics have played a central role in these investigations: first, the qualitative theory of ordinary differential equations and - closely related to that - the theory of dynamical systems; second, the theory of partial differential equations. Thus, in this same connection, mathematicians have successfully applied finite dimensional concepts and techniques, suitably modified, to the study of semigroups generated in infinite dimensional spaces by evolutionary partial differential equations.

In recent years several authors have developed and exploited this combination of finite dimensional and infinite dimensional techniques. Specifically, we cite the monographs by J. K. Hale [HA 2], R. Temam [TE 1], A. V. Babin and M. I. Vishik [B-V 2], and O. A. Ladyzhenskaya [LA 3]. This present book is in that same vein. In it we shall set forth the theory of asymptotic behavior for dynamical systems corresponding to parabolic equations and - in that connection - we will expound the theory of *global attractors*.

An important notion in these developments is that of *sectorial operator*, an idea studied by A. Friedman [FR 1] and extensively exploited by D. Henry [HE 1] More recently, H. Amann [AM 5], A. Lunardi [LU 1] and H. Tanabe [TA 2] have employed sectorial operators in their own investigations, which have emphasized the union of finite dimensional and infinite dimensional methods.

Of course, a major theme in this present book will be the use of sectorial operators in the study of parabolic problems. Also, in this present book, we will employ deep results in the *theory of interpolation* reported by H. Triebel [TR].

This book is divided into nine chapters, a brief description of which is as follows.

Chapter 1 is a review of several topics necessary for our later work; the choice of these topics is to ensure that this book is suitably self contained. Specifically, Chapter 1 summarizes some basic facts and definitions from the theory of Sobolev spaces, the theory of elliptic operators and sectorial operators - whose negatives generate analytic semigroups - and the theory of stability for dynamical systems.

Chapter 2 is devoted to equations of the form

$$(i) \qquad \begin{cases} \dot{u} + Au = F(u), \; t > 0, \\ u(0) = u_0. \end{cases}$$

where A is a sectorial operator and where F is a nonlinear function. The chapter itself deals with the local solvability of (i) in fractional order spaces X^α - domains of fractional powers A^α of A.

In Chapter 3 we continue our study of equations (i). Specifically, in that chapter, we investigate the extendibility of local solutions to the whole half line $[0, +\infty)$, and we investigate the smoothing action of the semigroup generated by (i). In connection with all this we introduce a condition - Condition (A_2) in Chapter 3 - sufficient for the global solvability of equations (i). We have found that this condition is very useful in several applications.

Chapter 4 is devoted to the construction of global attractors for equations (i). A slight strengthening of the condition (A_2) just mentioned leads to an *a priori* estimate - asymptotically independent of initial data - for the global solution constructed in Chapter 3. This estimate guarantees the existence of a global attractor for equations (i).

Chapter 5 provides a bridge between the abstract results of Chapters 2, 3, 4 and the applications of those results to several specific problems. The purpose is to translate the abstract formulations of Chapters 2-4 into the language of higher order parabolic problems in the setting of appropriately selected Sobolev spaces. In particular, Condition (A_2) - formulated in Chapter 3 - becomes an admissibility condition for the global solvability of equations (i). That condition - rendered as Restriction I in Chapter 5 - pertains to the growth rate of the nonlinear term F appearing in equations (i) and is closely related to the Nirenberg-Gagliardo estimate reported in Proposition 1.2.2 of Chapter 1.

In Chapter 6 we apply the abstract theory developed in earlier chapters to various problems arising in the physical and engineering sciences. These examples are arranged in an order of increasing complication. We begin with a new approach to the *method of invariant regions* for second order parabolic systems. From this we proceed to the construction of a global attractor for the *Cahn-Hilliard equation*. For this construction we do not require the usual growth condition imposed on the nonlinear term. Next, we construct a global attractor for the 2-*dimensional Navier-Stokes equation*. Finally in Chapter 6, we discuss the problem of constructing a global attractor for the *Cauchy problem* governed by second order equations whose spatial variable x belongs to R^n. In this last application there are special difficulties caused by a lack of compactness in the resolvent operator. We overcome these difficulties through an appropriate use of weighted Sobolev spaces.

Chapter 7 contains a further description of the solutions to (i). Based on the *backward uniqueness property* we justify invertibility of flows corresponding to examples introduced in Chapter 6. We also extend the notion of an X^α solution to the case of all $\alpha \geq 0$, thus providing additional information about the regularity of solutions.

In Chapter 8 we discuss some extensions of results obtained in Chapters 2-4 to problems with non-Lipschitz nonlinearities. We also investigate there local stability of stationary solutions of the n-dimensional Navier-Stokes equations. Furthermore in this same chapter we study the smoothness - in a classical sense - of the solutions we obtained in Chapters 2-4 for parabolic problems. In the same spirit we extend to spaces of Hölder functions our earlier results concerning the existence of a global attractor in spaces of fractional order. Finally in Chapter 8, we briefly examine degenerate parabolic equations; here our methods are borrowed from the theory of *monotone operators.*

Chapter 9, which closes this monograph, comprises a list of notations used in this book, a compendium of definitions relevant to the main body of the text, an abstract version of the maximum principle, and an iteration proof of an *a priori* L^∞ estimate for second order equations used in Chapter 6. Finally in Chapter 9, we compare various definitions of solutions to parabolic problems which can be found in basic monographs.

At the very end of this book, following Chapter 9, there appears a list of references. The books and articles listed therein indicate the wide background of the subject treated in this present book. Also, throughout this present book we cite relevant works from that list.

Although we consider this book to be only an introduction to the study of dynamics, nevertheless we believe that it will be of interest to mathematicians working in the areas of dynamical systems, evolutionary partial differential equations and their applications in mathematical physics and biology. We also believe that this book will be a suitable text for graduate students preparing themselves for studies in this field. Thus, we have tried to include most of the important results and proofs necessary for understanding the material being presented and its possible extensions.

This present book differs in two essential respects from the other works cited earlier in this preface. First, the modern semigroup approach to studying evolutionary systems - an approach expounded by D. Henry [HE 1] - plays a central role in our own exposition. Second, our approach to investigating the asymptotics of an evolutionary system parallels the approach taken by A. M. Lyapunov in his studies of ordinary differential equations, namely, for a given initial value problem, one constructs a local solution, then proves its unique extendibility to the whole half line $[0,+\infty)$, and then proceeds to the asymptotics. That approach dictates the general scheme underlying a large portion of this present book.

Here, at this juncture in the preface, we want to indicate some of the specific features of that general scheme.

If, in equations (i) above, the nonlinear term F acting from some X^α into X - with $\alpha \in [0,1)$ - is Lipschitz continuous on bounded sets, then the local solvability of (i) on $X^{\alpha+\varepsilon}$ - with $\varepsilon > 0$, $\alpha+\varepsilon < 1$ - follows. Next, for most equations coming from mathematical physics, there is available an *a priori estimate*

$$(ii) \qquad \|u(t)\|_Y \leq c(\|u_0\|_{X^\alpha}),\ t \geq 0,$$

for all putative X^α solutions $u(t)$, where Y is some space including $D(A)$. Often the etimate (ii) is obtained with the aid of an *energy functional.*

If the growth of $F : X^\alpha \to X$ on local solutions $u(t)$ is *sublinear relative to an a priori estimate* (ii), i.e., if

$$(iii) \qquad \|F(u(t))\|_X \le g(\|u(t)\|_Y)(1 + \|u(t)\|_{X^\alpha}),$$

where $g : R^+ \to R^+$ is nondecreasing, then each local $X^{\alpha+\varepsilon}$ solution ($\varepsilon > 0$, $\alpha + \varepsilon < 1$) can be uniquely extended to $[0, +\infty)$.

Finally, the *asymptotic independence* of the estimate (ii) relative to u_0, namely,

$$(iv) \qquad \limsup_{t\to+\infty} \|u(t)\|_Y \le const.,$$

ensures, as we shall see, the existence of a *global attractor* $\mathcal{A}$ for the semigroup $\{T(t)\}$ generated by (i) on $X^{\alpha+\varepsilon}$.

Much of this present book concerns the development of ideas implicit in the general scheme described above. Two remarks are in order.

First, in Chapter 3 we shall present a sufficient condition for the global extendibility of solutions. This condition was originally set forth in [FR 1] and was later treated in [AM 1], [WA 1], and [C-D 1]. As observed in [C-D 3], a slight modification of this same sufficient condition automatically guarantees the existence of a global attractor (cf. Theorem 4.1.1 below). Thus our study of asymptotics easily fits into the general scheme outlined above: local solvability, global extendibility, and existence of a global attractor.

Second, if $Y \subset X$, then the conditions (ii), (iii), and (iv) above are together necessary and sufficient for (i) to generate a point dissipative semigroup $\{T(t)\}$ on $X^{\alpha+\varepsilon}$ ($\varepsilon > 0$, $\alpha + \varepsilon < 1$) with bounded sets yielding bounded orbits. Hence, for any sectorial equation (i) in which the operator A has a compact resolvent, the conditions (ii), (iii), and (iv) are together necessary and sufficient for the existence of a global attractor.

Acknowledgments. We the authors wish to express our gratitude to Professor Jack K. Hale who has stimulated our research for many years. Especially, we would like to thank Professor Nathaniel Chafee for his significant contribution in improving a large portion of the text. We owe him a lot for this friendly help. We are also grateful to our friends Professor Nicolas D. Alikakos, Doctor Alexandre N. de Carvalho, and Doctor Andrzej W. Turski for their encouragement and for many fruitful discussions concerning various subjects in this book. Furthermore, we want to acknowledge all the support we have received from the Polish State Committee for Scientific Research (KBN) and also the United States Information Agency (USIA) and the Council for International Exchange of Scholars (CIES) under the Fulbright Program. This sponsorship has given the first author a fruitful visit at the Center for Dynamical Systems and Nonlinear Studies at the Georgia Institute of Technology in Atlanta, Georgia, USA.

Finally, we would like to express our gratitude to the anonymous referees whose

remarks and suggestions significantly improved the preliminary version of the manuscript. Also we would like to thank Doctor Alan Harvey and Mister Roger Astley, editors of mathematical sciences of the Cambridge University Press, for their professional assistance in the publication process.

Above all, for their patience and indulgence, the authors thank their families: Justyna and Tomasz Cholewa, and Maria and Paweł Dłotko. To them the authors lovingly dedicate this book.