CHAPTER 1

Preliminary Concepts

1.1. Elements of stability theory

A major portion of this book concerns the asymptotic behavior of solutions for nonlinear partial differential equations. Recent developments in the theory of dynamical systems show the unquestionable advantage of treating such solutions as an abstract flow on an appropriately selected phase space (e.g. [HE 1], [HA 2], [TE 1], [B-V 2]).

In this section we shall introduce some recently developed ideas in the theory of dynamical systems.[1] Our purpose here is to lay a foundation for analyzing and describing the long time dynamics of infinite dimensional differential equations.

1.1.1. Strongly continuous semigroups and stability of sets. We begin with the notion - basic for all our investigations - of a C^0 *semigroup* (*strongly continuous semigroup*).

Definition 1.1.1. *Let V be a metric space. A one parameter family $\{T(t)\}$ of maps $T(t): V \to V$, $t \geq 0$, is called a C^0 semigroup if*

(i) $T(0)$ is the identity map on V,
(ii) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,
(iii) the function

$$[0,\infty) \times V \ni (t,x) \to T(t)x \in V$$

is continuous at each point $(t,x) \in [0,\infty) \times V$.

It is known that for the semigroups of bounded linear operators in a Banach space X the condition (iii) holds if and only if, at any element $x \in X$,

$$T(t)x \to x \quad \text{when} \quad t \to 0^+,$$

which is basically a consequence of the Banach *uniform boundedness property*.

[1] Recall that a list of notations used in the book and a compendium of the basic definitions are presented at the beginning of Chapter 9.

For purposes of convenience, we shall forthwith introduce several concepts closely related to Definition 1.1.1 above. These concepts will frequently appear in our further considerations.

- The semigroup $\{T(t)\}$ is said to be *compact* if $T(t): V \to V$ is a compact map for each $t > 0$, i.e. each $T(t)$ takes bounded sets into precompact sets.
- The semigroup $\{T(t)\}$ is called *completely continuous* if it is compact and if for each bounded set $B \subset V$ and each number $t > 0$ the union $\bigcup_{s \in [0,t]} T(s)B$ is bounded in V.
- Let W_1, W_2 be two subsets of V. We say that W_2 is $\{T(t)\}$*-attracted* by W_1 if

$$d(T(t)W_2, W_1) \to 0 \quad \text{as} \quad t \to +\infty,$$

where, for each $t \geq 0$,

$$d(T(t)W_2, W_1) := \sup_{w_2 \in T(t)W_2} \inf_{w_1 \in W_1} dist_V(w_2, w_1).$$

Of course, it is true that the quantity $d(T(t)W_2, W_1)$ enters into the construction of the Hausdorff distance between the two sets $T(t)W_2$ and W_1. This is a matter we will not pursue here. However, we note that, roughly speaking, the number $d(T(t)W_2, W_1)$ measures how much the set $T(t)W_2$ lies outside the set W_1.

- Given any two subsets $W_1, W_2 \subset V$, we say that W_1 *absorbs* W_2 under $\{T(t)\}$ if there exists a number $t_0 \geq 0$ such that $T(t)W_2 \subset W_1$ for all $t \geq t_0$. Notice that W_1 attracts W_2 if and only if each open neighborhood $\mathcal{N}_{W_1}$ of W_1 in V absorbs W_2.
- An element $v \in V$ is called an *equilibrium point* for $\{T(t)\}$ if $T(t)v = v$ for all $t \geq 0$. Extending this notion, we say that a set $\mathcal{A} \subset V$ is $\{T(t)\}$*-invariant* if $T(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. Also, we will call $\mathcal{A} \subset V$ *positively* $\{T(t)\}$*-invariant* if $T(t)\mathcal{A} \subset \mathcal{A}$ for all $t \geq 0$.
- For any set $B \subset V$ the two sets $\gamma^+(B)$ and $\omega(B)$ defined by

$$\gamma^+(B) := \bigcup_{t \geq 0} T(t)B,$$
$$\omega(B) := \bigcap_{s \geq 0} cl_V \bigcup_{t \geq s} T(t)B$$

are called, respectively, the *positive orbit* and the *ω-limit set* of B. Thus, $\omega(B)$ consists of all points $v \in V$ for which there exist positive numbers $t_n \nearrow +\infty$ and points $v_n \in B$ with $T(t_n)v_n \to v$.
- For any point $v \in V$ we let S_v^- denote the set of all functions $\phi : (-\infty, 0] \to V$ such that $\phi(0) = v$ and such that $T(t)\phi(s) = \phi(t+s)$ whenever $-\infty < s \leq -t \leq 0$. Here we allow the possibility that S_v^- is empty, that S_v^- consists of exactly one element ϕ, or that S_v^- consists of more than one element ϕ. By a *negative orbit* through a given point $v \in V$ we mean any

set

$$\gamma_\phi^-(v) = \bigcup_{t\geq 0}\{\phi(-t)\},$$

where $\phi \in S_v^-$. Of course, there may or may not exist a nonempty negative orbit.

- For each point $v \in V$, a *complete orbit* through v is any set

$$\gamma_\phi(v) := \gamma^+(v) \cup \gamma_\phi^-(v),$$

where $\phi \in S_v^-$. Since we allow a negative orbit to be empty, we remark that a complete orbit $\gamma_\phi(v)$ is invariant if and only if its component $\gamma_\phi^-(v)$ is nonempty.

One of our principal preoccupations in this book will be the *stability properties* of invariant sets. This necessitates the following definition.

Definition 1.1.2. *Let $\mathcal{A} \subset V$ be nonempty and $\{T(t)\}$-invariant. We say that:*

*(i) $\mathcal{A}$ is **stable** if and only if for each open neighborhood U of $\mathcal{A}$ there exists an open neighborhood W of $\mathcal{A}$ such that $T(t)W \subset U$ for all $t \geq 0$;*
*(ii) $\mathcal{A}$ is **asymptotically stable** if and only if $\mathcal{A}$ is stable and attracts each point lying in some open neighborhood of $\mathcal{A}$;*
*(iii) $\mathcal{A}$ is **uniformly asymptotically stable** if and only if $\mathcal{A}$ is stable and attracts some open neighborhood of itself.*

When $\mathcal{A}$ is compact, in order to conclude that $\mathcal{A}$ is uniformly asymptotically stable, it suffices to show that $\mathcal{A}$ attracts some one of its open neighborhoods. More precisely:

Observation 1.1.1. *Let $\{T(t)\}$ be a C^0 semigroup in a metric space V. If $\mathcal{A}$ is compact and $\{T(t)\}$-invariant and if $\mathcal{A}$ attracts at least one of its own open neighborhoods, then $\mathcal{A}$ is stable.*

Proof. For the sake of argument suppose that $\mathcal{A}$ is not stable. Then there exists a neighborhood $\mathcal{N}_\mathcal{A}$ of $\mathcal{A}$ having the property that to each neighborhood W of $\mathcal{A}$ there correspond a point $w \in W$ and a number $t_w \geq 0$ such that $T(t_w)w \notin \mathcal{N}_\mathcal{A}$. From this and from the compactness of $\mathcal{A}$ there follows the existence of two sequences $t_n \to t_0 \in [0, +\infty]$ and $w_n \to w_0 \in \mathcal{A}$ such that

$$T(t_n)w_n \notin \mathcal{N}_\mathcal{A} \quad \text{for all} \quad n \in N. \tag{1.1.1}$$

In the case that $t_0 = +\infty$, (1.1.1) implies that there is no integer $n_0 > 0$ such that the bounded set $\{w_n,\ n \geq n_0\}$ is attracted to $\mathcal{A}$. This is absurd.

In the case that $t_0 < +\infty$, we have $T(t_n)w_n \to T(t_0)w_0$ and, since $\mathcal{A}$ is invariant, we must conclude that $T(t_0)w_0 \in \mathcal{A}$. Hence, the neighborhood $\mathcal{N}_\mathcal{A}$ contains all but finitely many elements of the set $\{T(t_n)w_n\}$. In view of (1.1.1) this also is absurd.

Thus in fact, $\mathcal{A}$ is stable. □

Compact, uniformly asymptotically stable subsets of the *phase space* are usually called *local attractors*. That is:

Definition 1.1.3. *A set $\mathcal{A} \subset V$ is called a* **local attractor** *for the semigroup $\{T(t)\}$ on V if and only if*

(i) $\mathcal{A}$ is nonempty, compact, and invariant with respect to $\{T(t)\}$;
(ii) $\mathcal{A}$ attracts some open neighborhood $\mathcal{N}_{\mathcal{A}}$ of $\mathcal{A}$.

For many infinite dimensional problems the notion of local attractor is, from a certain point of view, inadequate. Specifically, later in this book we will be seeking a characterization of long time behavior in terms of the dynamics of a suitably selected finite dimensional system. Such a characterization becomes possible if one can show the existence of a *global attractor* and then delineate its phase portrait. The notion of global attractor involves stability properties stronger than those formulated in Definition 1.1.3. Specifically:

Definition 1.1.4. *By a* **global attractor** *for $\{T(t)\}$ we mean a nonempty, compact, $\{T(t)\}$-invariant set $\mathcal{A} \subset V$ which attracts every bounded subset of V.*

Observation 1.1.2. *The global attractor - if it exists - is unique and also is maximal in the class of bounded invariant subsets of V.*

Proof. Indeed, let $\mathcal{A}, \mathcal{A}_1$ be compact invariant subsets of the phase space V, and suppose that $\mathcal{A}, \mathcal{A}_1$ each attract all the bounded subsets of V. Then, keeping in mind that $\mathcal{A}, \mathcal{A}_1$ are both bounded, we have

$$d(\mathcal{A}, \mathcal{A}_1) = d(T(t)\mathcal{A}, \mathcal{A}_1) \to 0 \quad \text{as} \quad t \to +\infty,$$
$$d(\mathcal{A}_1, \mathcal{A}) = d(T(t)\mathcal{A}_1, \mathcal{A}) \to 0 \quad \text{as} \quad t \to +\infty.$$

Consequently,

$$d(\mathcal{A}, \mathcal{A}_1) = d(\mathcal{A}_1, \mathcal{A}) = 0.$$

But both $\mathcal{A}$, $\mathcal{A}_1$ are closed. Therefore, $\mathcal{A} = \mathcal{A}_1$. One can also see that each bounded invariant subset of V must be contained in the global attractor. □

Observation 1.1.3. *The global attractor $\mathcal{A}$ is minimal in the class of all those closed bounded sets B in V which attract bounded sets.*

Proof. Indeed, consider any closed bounded set $B \subset V$ which attracts all bounded subsets of V. Then, arguing in a manner similar to the proof above for Observation 1.1.2, we obtain $d(\mathcal{A}, B) = 0$. From this it follows that $\mathcal{A} \subset B$. □

Before rendering our next observation we want to recall the notion of connectedness. Two sets $K, L \subset V$ are said to be separated if and only if

(1.1.2) $$L \cap cl_V K = K \cap cl_V L = \emptyset.$$

A set $S \subset V$ is called *connected* if and only if S cannot be decomposed into two separated sets K, L.

Observation 1.1.4. *The global attractor $\mathcal{A}$ is connected if and only if there exists a connected bounded set $B \subset V$ such that $\mathcal{A} \subset B$.*

Proof. If $\mathcal{A}$ is connected, then, taking $B = \mathcal{A}$, we trivially have a bounded connected set $B \subset V$ such that $\mathcal{A} \subset B$.

The converse assertion requires more argumentation. Specifically, suppose that B is a bounded connected subset V which contains $\mathcal{A}$. From the continuity of $T(t)$ it follows that the image $T(t)B$, $t \geq 0$, is connected.

By hypothesis, $\mathcal{A}$ is the global attractor for $\{T(t)\}$. Hence, $\mathcal{A}$ attracts every bounded set in V. It follows that, for each open neighborhood $\mathcal{N}_{\mathcal{A}}$ of $\mathcal{A}$, there exists a number $t_{\mathcal{N}_{\mathcal{A}}} > 0$ such that

$$\mathcal{A} \subset T(t)B \subset \mathcal{N}_{\mathcal{A}} \quad \text{for all} \quad t \geq t_{\mathcal{N}_{\mathcal{A}}}. \tag{1.1.3}$$

For the sake of argument, suppose that $\mathcal{A}$ is not connected, i.e., $\mathcal{A} = K \cup L$ where $K, L \subset V$ are nonempty and separated. We know that $\mathcal{A}$ is closed and we know that K, L satisfy (1.1.2). Hence, the sets K, L must each be closed. Thus, $\mathcal{A}$ decomposes into a sum of two disjoint nonempty closed sets K, L.

Since K, L are disjoint and closed, there must exist two open sets $\mathcal{U}_K, \mathcal{U}_L \subset V$ such that

$$\mathcal{U}_K \cap \mathcal{U}_L = \emptyset, \; K \subset \mathcal{U}_K, \; L \subset \mathcal{U}_L. \tag{1.1.4}$$

Let $\mathcal{N}_{\mathcal{A}}$ be the particular open neighborhood of $\mathcal{A}$ given by $\mathcal{N}_{\mathcal{A}} := \mathcal{U}_K \cup \mathcal{U}_L$. We know that $T(t)B$ is connected and that $\mathcal{U}_K, \mathcal{U}_L$ are separated. Therefore, with the aid of (1.1.3) we have either $\mathcal{A} = K \cup L \subset T(t)B \subset \mathcal{U}_K$ or $\mathcal{A} = K \cup L \subset T(t)B \subset \mathcal{U}_L$. But this last statement contradicts (1.1.4). The proof is complete. □

We remark that, if a metric space V is connected, then the global attractor $\mathcal{A}$ for a C^0 semigroup $T(t) : V \to V$, $t \geq 0$, is connected. However, this same statement is not necessarily true for discrete semigroups (see [GO-SA]).

Our goal now is to establish conditions guaranteeing the *existence* of the global attractor $\mathcal{A}$. To that end, we will introduce a class of semigroups having the property that, for each compact invariant set $B \subset V$, asymptotic stability is equivalent to uniform asymptotic stability.

1.1.2. Existence of a global attractor. In that which follows we will set forth conditions - formulated in [HA 2] - which guarantee the existence of a global attractor. These conditions relate to the notions of *dissipativeness* and *asymptotic smoothness* for $\{T(t)\}$.

Definition 1.1.5. *The semigroup $\{T(t)\}$ is called **point dissipative** if and only if there exists a nonempty, bounded set $B \subset V$ which attracts every point in V. The semigroup $\{T(t)\}$ is called **bounded dissipative** if and only if there exists a nonempty, bounded set $B \subset V$ which attracts every bounded subset of V.*

Definition 1.1.6. *The semigroup $\{T(t)\}$ is called **asymptotically smooth** if and only if each nonempty, closed, bounded, positively invariant set $W \subset V$ contains a nonempty, compact subset C which attracts W.*

Clearly, if the semigroup $\{T(t)\}$ has a global attractor $\mathcal{A}$ in V, then $\{T(t)\}$ must be dissipative in the sense of Definition 1.1.5. Furthermore, under the same conditions, $\{T(t)\}$ is of necessity asymptotically smooth. Indeed, consider any nonempty, closed, bounded set $W \subset V$ such that

$$T(t)W \subset W \quad \text{for} \quad t \geq 0. \tag{1.1.5}$$

Certainly, $\mathcal{A} \cap W$ is a closed subset of the compact set $\mathcal{A}$. Hence, $\mathcal{A} \cap W$ is compact. Moreover, since $\mathcal{A}$ attracts bounded sets, condition (1.1.5) ensures that $\mathcal{A} \cap W$ is nonempty and attracts W. Thus, as asserted, $\{T(t)\}$ is asymptotically smooth. We summarize these remarks as follows.

Observation 1.1.5. *If $\{T(t)\}$ is a C^0 semigroup on a metric space V and if $\{T(t)\}$ has a global attractor $\mathcal{A}$, then $\{T(t)\}$ is bounded dissipative and asymptotically smooth.*

At this moment we want to establish several important properties of *ω-limit sets for bounded sets* in the case that the semigroup $\{T(t)\}$ is asymptotically smooth (see [HA 2, Lemma 3.2.1]) .

Proposition 1.1.1. *Let $\{T(t)\}$ be a C^0 semigroup acting on a metric space V. If $\{T(t)\}$ is asymptotically smooth, if B is a nonempty subset of V, and if, for some number $t_B \geq 0$, the set*

$$\bigcup_{s \geq t_B} T(s)B$$

is bounded, then $\omega(B)$ is nonempty, compact, and invariant. Furthermore, $\omega(B)$ attracts B.

Proof. We start with the proof that $\omega(B)$ is compact. Since $\bigcup_{s \geq t_B} T(s)B$ is positively invariant and the maps $T(t)$ $(t \geq 0)$ are continuous, we have

$$T(t)cl_V \bigcup_{s \geq t_B} T(s)B \subset cl_V \bigcup_{s \geq t_B} T(s)B, \ t \geq 0.$$

Since the semigroup is asymptotically smooth there exists a nonempty, compact set $C \subset cl_V \bigcup_{s \geq t_B} T(s)B$ attracting $cl_V \bigcup_{s \geq t_B} T(s)B$. Hence C attracts B and, therefore,

$$\forall_{\substack{\mathcal{N}_C \text{ bounded, open} \\ \text{neighborhood of } C}} \ \exists_{\substack{\mathcal{N}'_C \text{ bounded, open} \\ \text{neighborhood of } C}} \ \exists_{t_{\mathcal{N}_C} \geq 0} \tag{1.1.6}$$
$$cl_V \bigcup_{t \geq t_{\mathcal{N}_C}} T(t)B \subset cl_V \mathcal{N}'_C \subset \mathcal{N}_C.$$

Condition (1.1.6) ensures that $\omega(B)$ is contained in each open neighborhood of C, which (since C is closed) implies the inclusion $\omega(B) \subset C$. Furthermore, using (1.1.6) and the compactness of C, one may show the existence of sequences $t_n \nearrow +\infty$ and $\{v_n\} \subset B$ for which $\{T(t_n)v_n\}$ is convergent. The set $\omega(B)$ is thus nonempty and, since $\omega(B)$ is closed and C is compact, $\omega(B)$ must be compact.

Next we shall prove that

$$d\Big(T(t)B,\omega(B)\Big)\to 0 \quad\text{when}\quad t\to+\infty. \tag{1.1.7}$$

Suppose that (1.1.7) is violated, i.e. there are an $\varepsilon>0$, a sequence $\{t_n\}$ increasing to infinity and a sequence $\{v_n\}\subset B$ such that

$$\inf_{y\in\omega(B)} dist_V\Big(T(t_n)v_n,y\Big)>\varepsilon,\ n\in N. \tag{1.1.8}$$

Then $\{T(t_n)v_n\}$ cannot have a convergent subsequence; otherwise such a limit point would belong to $\omega(B)$ and (1.1.8) would not be true. However, from (1.1.6) it follows that for any bounded, open neighborhood $\mathcal{N}_C$ of C almost all elements of $\{T(t_n)y_n\}$ are contained in $\mathcal{N}_C$. Since C is compact this allows one to choose from $\{T(t_n)y_n\}$ a convergent subsequence, which contradicts (1.1.8).

To prove that $\omega(B)$ is invariant note first, from the definition of $\omega(B)$ and continuity of $T(t)$, that the set $\omega(B)$ is positively invariant. Consider further any point $v_0\in\omega(B)$. Then $T(t_n)v_n\to v_0$ in V for some sequences $t_n\nearrow+\infty$ and $\{v_n\}\subset B$. Fix $t\geq 0$ and define

$$w_n := T(t_n-t)v_n,\ n\geq n_t.$$

Certainly

$$T(t)w_n\to v_0, \tag{1.1.9}$$

whereas, by (1.1.7) and the compactness of $\omega(B)$, there exists a subsequence $\{w_{n'}\}$ of $\{w_n\}$ convergent to some $w_0\in\omega(B)$. This and (1.1.9) ensure that $T(t)w_0=v_0$ which proves the inclusion $\omega(B)\subset T(t)\Big(\omega(B)\Big)$. The proof is complete. □

Corollary 1.1.1. *(see* [LA 3, Proposition 2.2]*) Let $\{T(t)\}$ be a C^0 semigroup on a metric space V, and suppose that $\{T(t)\}$ has a global attractor $\mathcal{A}$. Then*

(i) $\mathcal{A}$ is the union of the ω-limit sets of all bounded subsets of V,
(ii) $\mathcal{A}$ is the union of the ω-limit sets of all compact subsets of V,
(iii) $\mathcal{A}$ is the union of all bounded, invariant complete orbits through $v\in V$,
(iv) $\mathcal{A}$ is the union of all precompact, invariant complete orbits through $v\in V$.

Proof. First note from Observation 1.1.5 that $\{T(t)\}$ must be asymptotically smooth. Since $\mathcal{A}$ attracts bounded sets, for each bounded set $B\subset V$ there is a $t_B\geq 0$ such that $T(t_B)\Big(\gamma^+(B)\Big)$ is bounded. Therefore $\omega(B)$ is compact, invariant and, consequently, $\omega(B)\subset\mathcal{A}$. Moreover, $\omega(\mathcal{A})=\mathcal{A}$, and that completes the proof of (i). Similar reasoning establishes (ii). To prove (iii) (and (iv)), note first that, for each $v\in\mathcal{A}$, a negative orbit through v is nonempty. Hence, for each $v\in\mathcal{A}$ there exists an invariant, bounded, and precompact complete orbit through v. Therefore, $\mathcal{A}$ is contained in the union of such orbits. Obviously, each bounded, invariant complete orbit must lie within $\mathcal{A}$. Thus conditions (iii) (and (iv)) are established. □

In Definition 1.1.6 above we introduced the notion of asymptotically smooth semigroup and thus, in effect, singled out such semigroups for special attention. Our motive for doing this is embodied in the next observation.

Observation 1.1.6. *Let $\{T(t)\}$ be a C^0 semigroup in a metric space V and let $\mathcal{A} \subset V$ be a nonempty, compact, invariant set. If $\{T(t)\}$ is asymptotically smooth, then the following two conditions are equivalent:*

(a) $\mathcal{A}$ is asymptotically stable;

(b) $\mathcal{A}$ is uniformly asymptotically stable.

Proof. From Definition 1.1.2 it trivially follows that $(b) \Longrightarrow (a)$. Consequently, we need only prove that $(a) \Longrightarrow (b)$.

By hypothesis, $\mathcal{A}$ is asymptotically stable. Hence, there exists at least one bounded open neighborhood U of $\mathcal{A}$ such that $\mathcal{A}$ attracts points of U. Consider another open neighborhood U' of $\mathcal{A}$ such that $cl_V U' \subset \mathcal{U}$. Since $\mathcal{A}$ is stable, there must exist an open neighborhood $\mathcal{U}''$ of $\mathcal{A}$ such that

$$W := cl_V \bigcup_{t\geq 0} T(t)U'' \subset clU' \subset U.$$

Clearly W is nonempty, closed, bounded, and positively invariant. Therefore $\omega(W) \subset W$ and, by virtue of Proposition 1.1.1, $\omega(W)$ is compact and invariant. Also, $\omega(W)$ attracts W. Hence, in particular,

$$\omega(W) \text{ attracts } U''. \tag{1.1.10}$$

Also, since $\mathcal{A}$ is a compact, invariant subset of U'',

$$\mathcal{A} \subset \omega(W). \tag{1.1.11}$$

It is clear from the preceding considerations that $\mathcal{A}$ attracts points of $\omega(W)$. Thus, the continuity of $\{T(t)\}$ and the stability of $\mathcal{A}$ yield the statement that, for each open neighborhood $\mathcal{N}_{\mathcal{A}}$ of $\mathcal{A}$,

$$\exists_{\substack{\mathcal{N}'_{\mathcal{A}} \text{ an open} \\ \text{neighborhood of } \mathcal{A}}} \forall_{v\in\omega(W)} \exists_{t_v\geq 0} \exists_{\varepsilon_v>0} \quad T(t_v)B_V(v,\varepsilon_v) \subset \mathcal{N}'_{\mathcal{A}} \subset \gamma^+(\mathcal{N}'_{\mathcal{A}}) \subset \mathcal{N}_{\mathcal{A}}, \tag{1.1.12}$$

where $B_V(v,\varepsilon_v)$ denotes the open ball in V centered at v and having radius ε_v. Let $\mathcal{N}_{\mathcal{A}}$ be fixed. From the compactness of $\omega(W)$ there follows

$$\exists_{k\in N} \exists_{v_1,\dots,v_k\in\omega(W)} \ \omega(W) \subset B_V(v_1,\varepsilon_{v_1}) \cup \ldots \cup B_V(v_k,\varepsilon_{v_k}). \tag{1.1.13}$$

Introduce

$$t_{\max} := \max\{t_{v_1},\dots,t_{v_k}\}.$$

The assertions (1.1.13), (1.1.12) and the invariance of $\omega(W)$ imply

$$\omega(W) = T(t_{\max})\big(\omega(W)\big) \subset T(t_{\max})\left(\bigcup_{j=1}^{k} B_V(v_j,\varepsilon_{v_j})\right) = \bigcup_{j=1}^{k} T(t_{\max})B_V(v_j,\varepsilon_{v_j})$$

$$= \bigcup_{j=1}^{k} T(t_{\max}-t_{v_j})T(t_{v_j})B_V(v_j,\varepsilon_{v_j}) \subset \gamma^+(\mathcal{N}'_{\mathcal{A}}) \subset \mathcal{N}_{\mathcal{A}}.$$

All of this implies that $\omega(W)$ is contained in each open neighborhood of $\mathcal{A}$. Since $\mathcal{A}$ is compact, we obtain $\omega(W) \subset \mathcal{A}$ which, by virtue of (1.1.11), leads us to the equality $\omega(W) = \mathcal{A}$. Taking into account (1.1.10) we see that $\mathcal{A}$ attracts its open neighborhood U''. Thus, $\mathcal{A}$ is uniformly asymptotically stable and the proof is complete. □

The theorem stated below, due to J. K. Hale [HA 2], gives sufficient conditions for the existence of a global attractor.

Theorem 1.1.2. *Let $\{T(t)\}$ be a C^0 semigroup on a metric space V. If $\{T(t)\}$ is point dissipative, asymptotically smooth, and keeps orbits of bounded sets bounded, then $\{T(t)\}$ has a global attractor in V.*

Proof. The proof proceeds in two steps.

Step 1. We shall first show the existence of a bounded set $\mathcal{O} \subset V$ such that each compact set $C \subset V$ has an open neighborhood $\mathcal{N}_C$ which is absorbed by $\mathcal{O}$.

By hypothesis there exists a nonempty bounded set $W_0 \subset V$ which attracts points of V. Let $\mathcal{N}_{W_0}$ be any bounded open neighborhood of W_0. Using the continuity of $\{T(t)\}$ and the fact that $\mathcal{N}_{W_0}$ absorbs points of V we conclude that

(1.1.14) $$\forall_{v\in V}\ \exists_{\tau_v\geq 0}\ \exists_{B_V(v,\varepsilon_v)}\ T(\tau_v)B_V(v,\varepsilon_v) \subset \mathcal{N}_{W_0},$$

where $B_V(v,\varepsilon_v) \subset V$ denotes the open ball in V centered at v and having radius ε_v. Next, choose $t_{\mathcal{N}_{W_0}} \geq 0$ such that

$$\mathcal{O} := \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T(t)(\mathcal{N}_{W_0})$$

is bounded. By our assumptions, $\mathcal{O}$ is positively invariant and absorbs points of V. Moreover, from (1.1.14) we observe that

(1.1.15) $$\forall_{v\in V}\ \exists_{t_v:=\tau_v+t_{\mathcal{N}_{W_0}}\geq 0}\ \exists_{B_V(v,\varepsilon_v)}\ T(t)B_V(v,\varepsilon_v) \subset \mathcal{O} \ \text{ for } \ t \geq t_v.$$

Consider now any compact set $C \subset V$. Certainly $C \subset \bigcup_{v\in C} B_V(v,\varepsilon_v)$ so that, from the compactness of C,

(1.1.16) $$C \subset B_V(v_1,\varepsilon_{v_1}) \cup \ldots \cup B_V(v_k,\varepsilon_{v_k}) =: \mathcal{N}_C$$

for some $k \in N$ and $v_1,\ldots,v_k \in C$. With the aid of (1.1.15), (1.1.16) we obtain

$$T(t)C \subset T(t)\mathcal{N}_C = \bigcup_{j=1}^{k} T(t)\left(B_V(v_j,\varepsilon_{v_j})\right) \subset \mathcal{O} \ \text{ for } \ t \geq \max\{t_{v_1},\ldots,t_{v_k}\}.$$

Step 2. We shall now construct a compact invariant set $\mathcal{A}$ which attracts each bounded subset of V.

Let $B \subset V$ be a bounded set. By our assumptions Proposition 1.1.1 ensures that $\omega(B)$ is compact and attracts B, i.e.

$$\forall_{\substack{\mathcal{N}_{\omega(B)} \text{ an open} \\ \text{neighborhood of } \omega(B)}} \exists_{t_B \geq 0}\, T(t)B \subset \mathcal{N}_{\omega(B)} \text{ for } t \geq t_B. \tag{1.1.17}$$

However, as shown in Step 1, there exists some open neighborhood $\mathcal{N}_{\omega(B)}$ of $\omega(B)$ absorbed by $\mathcal{O}$. From this result and condition (1.1.17) we obtain

$$\forall_{\substack{B \subset V \\ B \text{ bounded}}} \exists_{\tau_B \geq 0}\, T(t)B \subset \mathcal{O} \text{ for } t \geq \tau_B. \tag{1.1.18}$$

Let $\mathcal{A} := \omega(\mathcal{O})$. Using Proposition 1.1.1 again we find that $\mathcal{A}$ is compact, is invariant and attracts $\mathcal{O}$. Furthermore $\mathcal{A}$ attracts bounded subsets of V since, as shown in (1.1.18), bounded sets are absorbed by $\mathcal{O}$. The proof is complete. □

Remark 1.1.1. This theorem was introduced in [HA 2] with the comment that the result is somewhat surprising inasmuch as, in the presence of a weak form of dissipation, orbits of bounded sets converge uniformly to some compact set. With regard to applications, this last comment is particularly significant. Indeed, it is much easier to obtain pointwise estimates on trajectories than it is to establish the existence of an absorbing set.

Remark 1.1.2. As a result of Theorem 1.1.2 and Observation 1.1.5 we conclude that *if $\{T(t)\}$ is a C^0 semigroup on a metric space V and if, under $\{T(t)\}$, the orbits of bounded sets are bounded, then $\{T(t)\}$ has a global attractor in V if and only if $\{T(t)\}$ is point dissipative and asymptotically smooth.* In general the boundedness of orbits for bounded sets is not necessary for the existence of a global attractor, although this is true for a large number of systems (e.g. compact semigroups corresponding to sectorial equations discussed in Chapter 3 have this property). From the point of view of Theorem 1.1.2, such an assumption may be weakened, since what we actually used in the proof was the property that

$$\forall_{\substack{B \subset V \\ B \text{ bounded}}} \exists_{t_B \geq 0} \bigcup_{t \geq t_B} T(t)B \text{ is bounded in V.} \tag{1.1.19}$$

On the basis of this remark we have (see [LA 3, Theorem 3.4])

Corollary 1.1.3. *A C^0 semigroup $\{T(t)\}$ on a metric space V has a global attractor if and only if $\{T(t)\}$ is point dissipative, is asymptotically smooth, and satisfies the condition* (1.1.19).

Remark 1.1.3. Looking again at the proof of Theorem 1.1.2 let us also note that instead of the point dissipativeness of $\{T(t)\}$ we might assume a weaker condition:

$$\exists_{\substack{B \subset V \\ B \text{ bounded}}} \forall_{u_0 \in V} \exists_{t_{u_0} \geq 0}\, T(t_{u_0})u_0 \in B \tag{1.1.20}$$

(i.e. $\gamma^+(u_0) \cap B \neq \emptyset$ for each $u_0 \in V$). Corollary 1.1.3 then becomes