1

Computerized Tomography, X-rays, and the Radon Transform

1.1 Introduction

The purpose of this chapter is to give an informal introduction to the subject of tomography. There are very few mathematical requirements for this chapter, so readers who are not specialists in the field, indeed who are not mathematicians or scientists, should find this material accessible and interesting. Specialists will find a graphic and intuitive presentation of the Radon transform and its approximate inversion.

Tomography is concerned with solving problems such as the following. Suppose that we are given an object but can only see its surface. Could we determine the nature of the object without cutting it open? In 1917 an Austrian mathematician named Johann Radon showed that this could be done provided the total density of every line through the object were known.[1] We can think of the density of an object at a specific point as the amount of material comprising the object at that point. The total density along a line is the sum of the individual densities or amounts of material.

In 1895 Wilhelm Roengten discovered x-rays, a property of which is their determining of the total density of an object along their line of travel. For this reason, mathematicians call the total density an ***x-ray projection***. It is immaterial whether the x-ray projection was obtained via x-rays or by some other method; we still call the resulting total density an x-ray projection.

Combining Roengten's x-rays with Radon's idea gives a way of determining an unknown object without cutting it open. We call this process ***tomography***.

Tomography can be applied to any object for which we can determine the x-ray projections either by actual x-rays or some other method. Tomography is used to investigate the interior structure of the following objects: the human body, rocket motors, rocks, the sun (microwaves were used here rather than x-rays), snow packs on the Alps, and violins and other bowed instruments. This list could be expanded to hundreds of objects. In this chapter we will see how tomography can be used to obtain detailed information about the human brain from its x-ray projections.

[1] Johann Radon (1887–1956) published the first discussion and solution of a tomographic problem (see reference [508] in the bibliography).

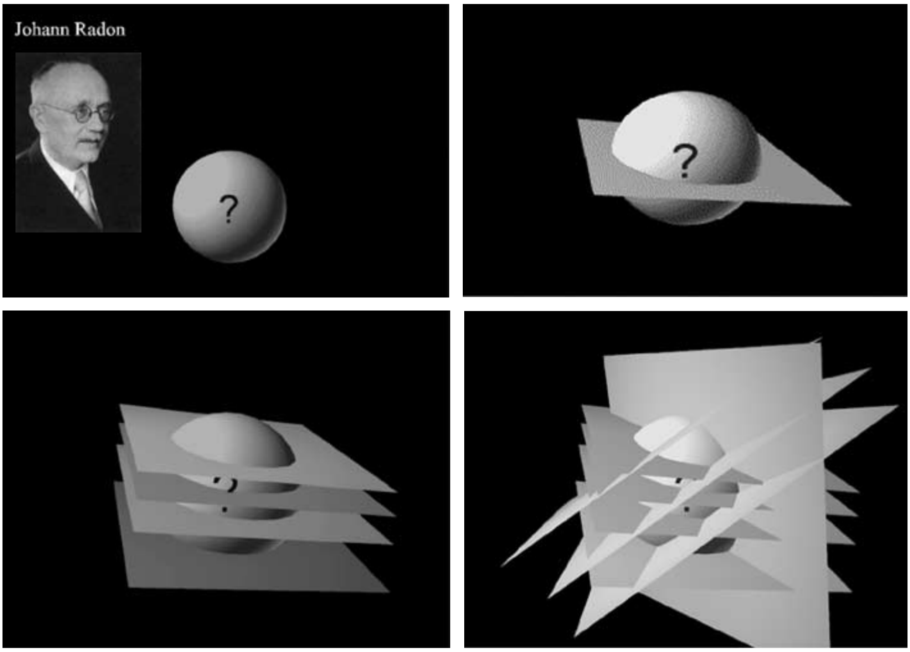


Figure 1.1. Johann Radon tries to figure out what is inside the sphere.

Allan M. Cormack and Godfrey N. Hounsfield shared the 1979 Nobel Prize in Physiology and Medicine for their contributions to the medical applications of tomography. The reference to snow packs comes from Cormack's Nobel prize lecture in Stockholm in 1979 (compare [102]).[2] Cormack remarked that the publication of his ideas on tomography took place in 1963 and 1964 and that "*There was virtually no response. The most interesting request for a reprint came from the Swiss Centre for Avalanche Research. The method would work for deposits of snow on mountains if one could get either the detector or the source into the mountain under the snow!*"

Radon not only showed how to determine a plane object from lines, but he also showed how to determine a solid object by using planes. We can visualize the discussion up to this point. In figure 1.1 Johann Radon is pondering what is inside the spherical object. In the next scene he decides to compute the total density on a single plane through the sphere. He knows that this is not enough information to determine the object, so he successively intersects with more and more planes. When he has collected the densities on all planes, then he is able to determine the object. How this may be done by using lines through a two-dimensional object is the subject of the remainder of this chapter. You do not need much background in mathematics to read this chapter – some knowledge about triangles and the ability to read a graph is really all that is required.

[2] Numbers in square brackets correspond to the list of references at the end of the book. For example, [102] refers to the article by Cormack that is listed in the references section.

1.2 Computerized Tomography (CT) and Mathematical Tomography – "Now, suddenly, the fog had cleared"

The Greek word $\tau o \mu o \sigma$ (tomos), meaning slice, is the source of the term ***tomography***. This term was first used in diagnostic medicine. Since the discovery of x-rays by Roentgen, diagnosticians have attempted to produce images of human organs without the blurring and overlap of tissue that occurs in traditional x-ray pictures, such as the x-ray of the skull in the accompanying figure.

Courtesy of Ass. Prof. Dr. Mircea-Constantin Sora, MD, Ph. D., Medical University of Vienna.

We will see that tomography can produce much more detailed pictures from x-ray data. The reference to the fog clearing in the title of this section is from the presentation speech for the 1979 Nobel prize for Physiology or Medicine which was awarded, jointly to A. M. Cormack and G. N. Hounsfield in 1979. The presentation speech containing the preceding quotation was delivered by Professor Torgny Greitz of the Karolinska Medico-Chirurgical Institute and it is interesting to read the excerpts from this speech in Section 1.10.1.

Computerized Tomography, also known as ***CT***, refers to the actual process of producing a detailed picture of the interior of an organism by using x-rays. ***Mathematical tomography*** refers to the mathematical process by which the picture is obtained. Computerized tomography is accomplished by designing a machine consisting of x-ray sources and x-ray detectors combined with a computer. The computer uses an algorithm adapted from the field of mathematical tomography to combine the data obtained from the x-rays into a detailed picture of the organs and tissue in a specific slice of a

Figure 1.2. A typical CT scanner. This one is manufactured by the Siemens Corporation. Courtesy of the Siemens corporation.

patient's body. This type of machine is called a ***CT scanner***.[3] A CT scanner can produce a clear and detailed image, called a ***tomogram***, of the interior of a human body. This is done without cutting open the body, merely by sending x-rays through the tissue in question. How this is done is explained later in this chapter.

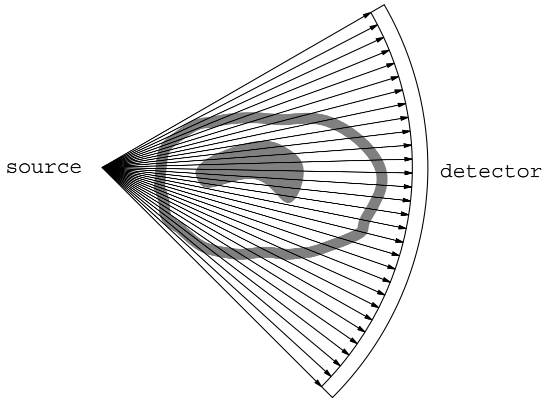
Some forms of tomography were used in diagnostic medicine long before computers were invented (see Section 1.10). A typical method attempted to visualize a section (slice) of a body by blurring out all the x-rays except those in the focal plane of the desired slice. Early CT scanners also concentrated on a single slice of the body. This attention on a slice (from $\tau o \mu o \sigma$) explains the origin of the word tomography in medicine. The desire was to visualize a sliced human body without actually slicing it. In mathematical tomography the slicing refers to the lines or planes that slice through the object of interest.

Figure 1.2 is a picture of a typical CT scanner.

The circular ring in the CT scanner emits x-rays from a source on one side. These x-rays are detected at the opposite side. The ring rotates so that x-rays can be beamed, in any direction, through a specific slice of the patient's body. Here is a diagram of how this operates.

[3] A CT scanner is also referred to as a CAT scanner, which is derived from "computer-assisted tomography," whereas CT derives from "computerized tomography." The preferred term is CT scanner, although CAT scanner is informally and ubiquitously used.

There is the story of the man who brought his sick dog to the veterinarian. Upon examination, the veterinarian pronounced the dog dead. The distraught owner replied: "That is impossible, I know my dog is listless, but certainly not dead. Is there not a more definitive test that you can do?" "Very well," replied the veterinarian, who immediately summoned a black-and-white cat. The cat proceeded to examine the dog. First, the cat only sniffed around the dog who exhibited no reaction. Then the cat hissed at the dog and finally clawed it, all without reaction from the dog. The owner finally said, "I suppose you are right, my dog is dead. How much do I owe you?" The veterinarian replied, "That will be $300." The owner retorted. "Three hundred dollars to tell me my dog is dead! That is outrageous! Why is it so much?" "The veterinarian replied, "It is $100 for the examination and $200 for the CAT scan."



This mode of scanning is called ***fan beam geometry*** for obvious reasons.[4] CT scanners that use fan beam geometry are called ***fan beam scanners***. In this mode x-rays are generated at the source. They form a beam in the shape of a fan and are observed at the detector after passing through the body. In this way the total density along every line emanating from the source can be computed. By rotating the apparatus, the source and detectors move to new positions. In this way the total density along every line intersecting the body can be determined. These total densities are the input data to an algorithm that reconstructs a picture of the organs and tissue in this slice. Later we will show how these total densities can be used to reconstruct an image of the tissue. Meanwhile, see figure 1.3 for a comparison of a traditional x-ray of the head and a tomogram of a section of a human brain. Note the lack of detail of the

Figure 1.3. (*Left*) traditional x-ray. (*Right*) Tomographic reconstruction of a brain section. Image on the left courtesy of Ass. Prof. Dr. Mircea-Constantin Sora, MD, Ph. D., Medical University of Vienna.

[4] The analogous situation in three dimensions is called *cone beam geometry*.

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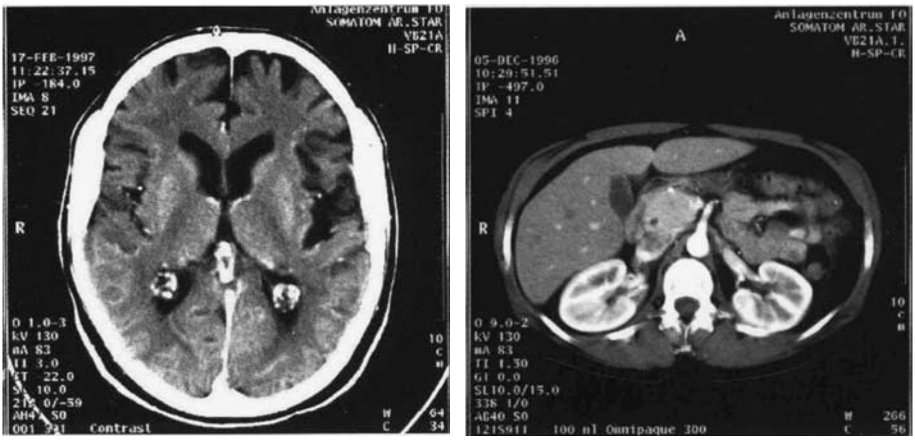
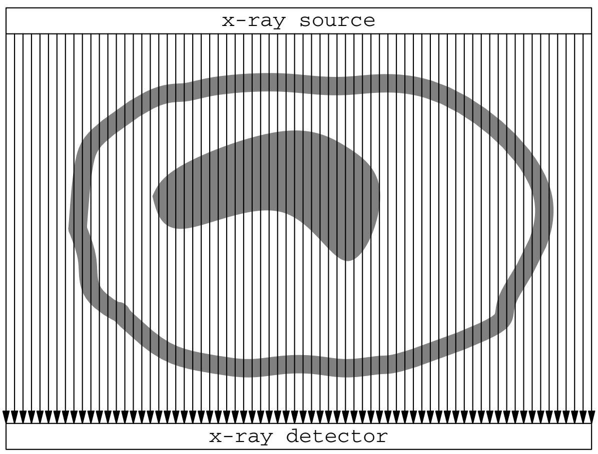


Figure 1.4. CT images. The left image is a cross section of a human brain. The right image is a cross section of a human abdomen. Courtesy of the Siemens corporation.

brain in the traditional x-ray compared with the fine detail in the tomogram. Another set of tomograms is in figure 1.4.

When an x-ray beam is sent through tissue, it experiences more attenuation by heavier tissue than by lighter tissue. For example, the skull is about twice as dense as the gray matter of the brain. Therefore, x-rays are more likely to be absorbed or scattered when passing through the skull than when passing through gray matter. Although there are some subtleties with this idea, we can make a working assumption that sending an x-ray beam through an object determines the total density of the object on the line intersected by the x-ray.

Before continuing we should mention that older CT machines used a parallel beam geometry. Their mode of operation is illustrated by the following figure.

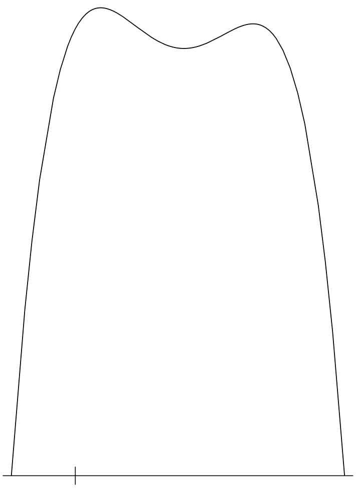


It is much more efficient to use fan beam geometry and most modern CT scanners use this method. In either method we can obtain information about the density of the object along any line, provided that the scanner is free to rotate through 180°. It is simpler to describe the mathematics for parallel beam scanners and from now on we will do so. The method of reconstructing images via parallel beam geometry can be applied to fan beam scanners because, as we already noted, information about any line intersecting the object can be obtained in either geometry. However, the computational effort in reorganizing the fan beam data into parallel beam data is substantial. There are algorithms that use the fan beam data directly and these will be described in a later chapter. The efficiency of fan beam algorithms together with the efficiency of the fan beam scanning geometry make current CT scanners much faster than older ones. The latest generation of CT scanners emit x-rays along a helical path and reconstruct three-dimensional pictures.

1.3 Objects and Functions

Tomography is an example of a classical mathematical problem: determine an unknown quantity when some given information is provided. The unknown quantity might be a real number x, which is in some relation to some known real numbers, for example, $5x + 1 = 7$. In other situations the unknown quantity might be a function with some given information about its behavior. For example, determine the unknown position of a particle given its acceleration, its initial position, and its initial velocity. In a tomographic problem the given information is a set of x-ray projections of an unknown object. The solution is an exact or approximate representation of the unknown object obtained by mathematical manipulation of the known x-ray data.

At this point we need a precise definition of the term "object." Let us take a simple object, say a two-dimensional image of the profile of a mountain:



To specify this object mathematically, all we need to know is the height of each point of the curve above the ground.[5] Such a specification is called a function. Many

[5] In this example the exact height is given by $-\frac{1}{4}x^4 - \frac{3}{8}x^3 + \frac{1}{2}x^2 + \frac{3}{8}x + \frac{27}{4}$, where the unit for x is denoted by the mark on the ground line in the picture.

two-dimensional shapes can be represented by functions in this way. In general, a ***function*** is a rule that uniquely assigns a value to each element of a given set. The given set is called the ***domain*** of the function. In this chapter we assume that the value assigned to an element of the domain is a real number.

An example of a function is given by the rule f which assigns $\frac{1}{x}$ to every nonzero real number x. Here the domain is the set of non-zero real numbers. We can define this rule by writing the equation

$$f(x) = \frac{1}{x}$$

Note that we use a letter, in this case f, to represent the rule or function. Then, for each x in the domain, $f(x)$ represents the quantity obtained by applying this rule to x. The symbol $f(x)$ is read as "the value of f at x" or, in brief, "f of x." It is important to conceive of a function as a single object and not to confuse f with $f(x)$. However, sometimes we are sloppy and use $f(x)$ to denote the function f, even though $f(x)$ actually is a real number representing the *value* of the function f at x.

It is not necessary for the domain of a function to consist of numbers. For example, we can consider the function h, which assigns to every horse its weight in kilograms. In this case the domain is the set of all horses.

In the profile of the mountain, the domain was one-dimensional (the ground line). The function f contained all the information needed to describe the mountain's profile. Although the profile of the mountain is two-dimensional, the amount of information needed to determine the profile is only one-dimensional. This is because the function describing the elevation is of the form $f(x)$, where x is a single variable that moves along a straight line.

In general, a two-dimensional object will require two variables to be completely determined. A general point in the plane is uniquely determined by the coordinates (x, y). Therefore, we can treat more general two-dimensional objects by specifying a function of two variables: $f = f(x, y)$.

Figure 1.5 is an image of an abdominal section of a human patient. The density of the tissue at each point is depicted by the amount of gray at that point. The black points are the most "gray" and represent zero density. The actual tissue has varying density. The highest-density points are the least "gray" (white) and denote bone. The other tissue is less dense and is represented by various shades of gray. To describe this picture, all we need to know is the relative amount of gray to put at each point. This amount can be specified by a real number. Therefore, for all practical purposes, this image can be represented by a function f of two variables: $f(x, y)$ represents the amount of gray to place at the point (x, y) to create this picture.

An object can thus be viewed as a function of two real variables x and y, because we are uniquely assigning a real number (a gray value) to every point (x, y) of the plane. Conversely, if we are given a function of two variables, then we can view the associated object by using the value of the function at every point as a gray value. So from the mathematical point of view, there is no difference between a function of two variables and a two-dimensional object. For this reason we use the term "object" and

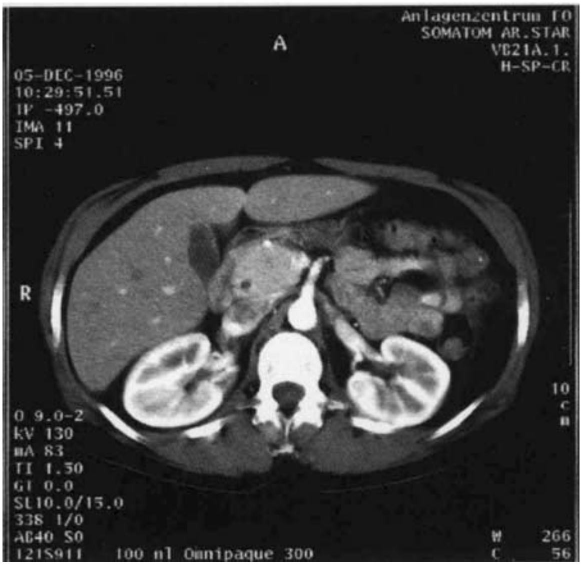
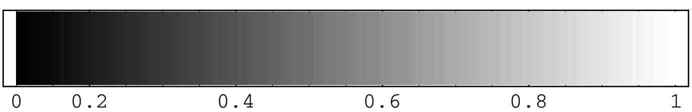


Figure 1.5. Tomogram of a human abdominal section. Courtesy of the Siemens Corporation.

the term “function” interchangeably. Also, for this reason we use letters like f and g to represent objects.

There are two main ways to exhibit the value of a function at a point. The first way is called a ***density plot*** and it attaches a gray level to each point in the domain. An example of a density plot is the tomogram in figure 1.5. Each gray level represents a specific real number, the lowest values of the function shown in black and the highest values shown in white.

The gray scale presented in the next figure shows how any real number between 0 and 1 can be represented as a gray level. It is not a function, it only serves to establish the correspondence of gray levels to a range of real numbers. The gray scale plays the same role as the x axis in a graph – it shows how we represent real numbers. One purpose of the gray scale is to establish the range of numbers used in the graph of the object. The range does not have to be from 0 to 1. It could be from any real number a to any larger real number b. However, the smallest value will always be represented as black and the largest as white. After this example, we will not be fussy about the actual range of values, so we will present objects without the accompanying gray scale.



The other way of graphing an object is to place a point of height $f(x, y)$ above the location (x, y) in the plane. This type of view is called the ***graph*** (of the object). An example of a graph of an object may be found in figure 1.6.

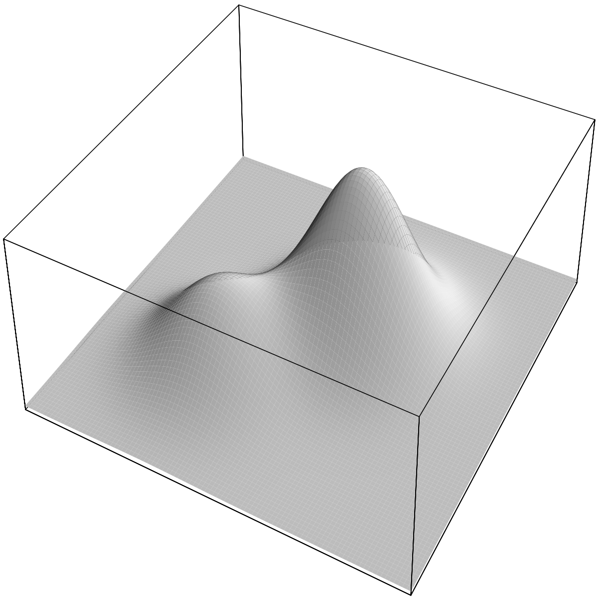


Figure 1.6. Graph of a function representing a mountain.

Simple objects may be represented by functions that take the value 1, represented by white, at all points that lie on the object, and that take the value zero, represented by black, elsewhere. Therefore, their density graphs will be exactly the shape of the object. To avoid becoming overly wordy let us agree that when we use a term such as triangular object we really mean the function that is 1 on the triangle and zero elsewhere.

Here is the density plot of a square object.