Introduction

What is “Analysis on Fractals”? Why is it interesting?

To answer those questions, we need to go back to the history of fractals.

Many examples of fractals, like the Sierpinski gasket, the Koch curve and the Cantor set, were already known to mathematicians early in the twentieth century. Those sets were originally pathological (or exceptional) counterexamples. For instance, the Koch curve (see Figure 0.1) is an example of a compact curve with infinite length and the Cantor set is an example of an uncountable perfect set with zero Lebesgue measure. Consequently, they were thought of as purely mathematical objects. In fact, they attracted much interest in harmonic analysis in connection with Fourier transform, and in geometric measure theory. There were extensive works started in the early twentieth century by Wiener, Winter, Erdös, Hausdorff, Besicovich and so on. See [181], [32] and [124]. These sets, however, had never been associated with any objects in nature.

This situation had not changed until Mandelbrot proposed the notion of fractals in the 1970s. In [122, 123] he claimed that many objects in nature are not collections of smooth components. As evidence, using the experiments by Richardson, he showed that some coast lines were not smooth curves but curves which have infinite length like the Koch curve. Choosing

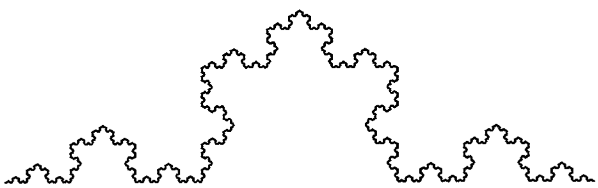

Fig. 0.1. Koch curve

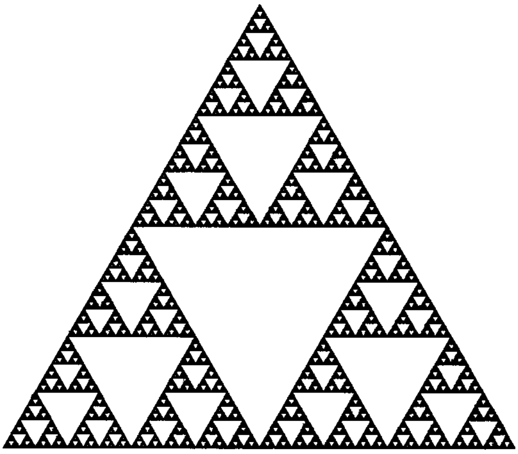

Fig. 0.2. Sierpinski gasket

words more carefully and accurately, we need to say that some coast lines should be modeled by curves with infinite length rather than (compositions of) smooth curves.

Mandelbrot coined this revolutionary idea and introduced the notion of fractals as a new class of mathematical objects which represent nature. The importance of his proposal was soon recognized in many areas of science, for example, physics, chemistry and biology. In mathematics, a new area called fractal geometry developed quickly on the foundation of geometric measure theory, harmonic analysis, dynamical systems and ergodic theory. Fractal geometry treats the properties of (fractal) sets and measures on them, like the Hausdorff dimension and the Hausdorff measure. From the viewpoint of applications, it concerns the static aspects of the objects in nature.

How about the dynamical aspects? There occur (physical) phenomena on those objects modeled by fractals. How can we describe them? More precisely, how does heat diffuse on fractals and how does a material with a fractal structure vibrate? To give an answer to these questions, we need a theory of "analysis on fractals". For example, on a domain in $\mathbb{R}^n$, diffusion of heat is described by the heat (or diffusion) equation,

$$\frac{\partial u}{\partial t} = \Delta u,$$

where $u = u(t,x)$, t is time, x is position and Δ is the Laplacian defined by $\sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$. If our domain is a fractal, we need to know what the

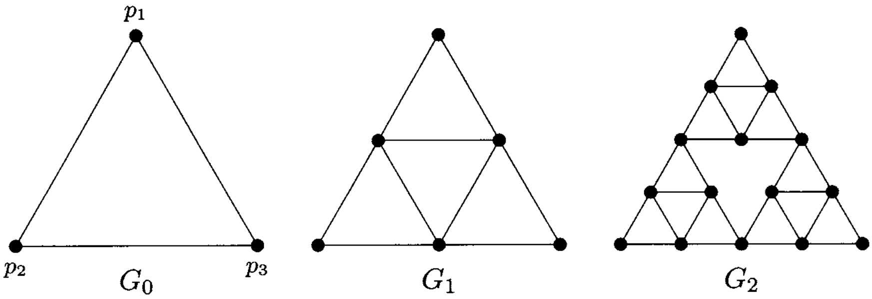

Fig. 0.3. Approximation of the Sierpinski gasket by graphs G_m

"Laplacian" on it is. This problem contains somewhat contradictory factors. Since fractals like the Sierpinski gasket and the Koch curve do not have any smooth structures, to define differential operators like the Laplacian is not possible from the classical viewpoint of analysis. To overcome such a difficulty is a new challenge in mathematics and this is what analysis on fractals is about.

During the 1970s and 1980s, physicists tried to describe phenomena on fractals. They succeeded in calculating some of the physical characteristics of fractals, for example, the spectral exponent, which should describe the distribution of the eigenstates. (See, for example, [118] and [75] for reviews of studies in physics.) However they did not know how to define "Laplacians" on fractals. See Note and References of Chapter 4 for details.

Motivated by studies in physics, Kusuoka [106] and Goldstein [51] independently took the first step in the mathematical development. They constructed "Brownian motion" on the Sierpinski gasket. Their method of construction is now called the probabilistic approach. First they considered a sequence of random walks on the graphs which approximate the Sierpinski gasket and showed that by taking a certain scaling factor, those random walks converged to a diffusion process on the Sierpinski gasket. To be more precise, let us define the Sierpinski gasket. Let $\{p_1, p_2, p_3\}$ be a set of vertices of an equilateral triangle in $\mathbb{C}$. Define $f_i(z) = (z - p_i)/2 + p_i$ for $i = 1, 2, 3$. Then The Sierpinski gasket K is the unique non-empty compact subset K of $\mathbb{R}$ that satisfies

$$K = f_1(K) \cup f_2(K) \cup f_3(K).$$

See Figure 0.2. Let $V_0 = \{p_1, p_2, p_3\}$. Define a sequence of finite sets $\{V_m\}_{m \geq 0}$ inductively by $V_{m+1} = f_1(V_m) \cup f_2(V_m) \cup f_3(V_m)$. Then we have the natural graph G_m whose set of vertices is V_m. (See Figure 0.3.)

For $p \in V_m$, let $V_{m,p}$ be the collection of the direct neighbors of p in V_m. Observe that $\#(V_{m,p}) = 4$ if $p \notin V_0$ and $\#(V_{m,p}) = 2$ if $p \in V_0$, where $\#(A)$ is the number of elements in a set A. Let $X_t^{(m)}$ be the simple random walk on G_m. (This means that if a particle is at p at time t, it will move to one of the direct neighbors with the probability $\#(V_{m,p})^{-1}$ at time $t+1$.) What Kusuoka and Goldstein proved was that

$$X_{5^{-m}t}^{(m)} \to X_t$$

as $m \to \infty$, where X_t was a diffusion process, called Brownian motion, on the Sierpinski gasket. In this probabilistic approach, a Laplacian is the infinitesimal generator of the semigroup which is associated with the diffusion process.

Barlow and Perkins [20] followed the probabilistic approach and obtained an Aronson-type estimate of the heat kernel associated with Brownian motion on the Sierpinski gasket. See Notes and References of Chapter 5. Then, in [116], Lindstrøm extended this construction of Brownian motion to nested fractals, which is a class of finitely ramified self-similar sets with strong symmetry. See 3.8 for the definition of nested fractals. (Roughly speaking, finitely ramified self-similar sets are the self-similar sets which become disconnected if one removes a finite number of points. See 1.3 for details.)

On the other hand, in [82], a direct definition of the Laplacian on the Sierpinski gasket was proposed. Under this direct definition, one could describe the structures of harmonic functions, Green's function and solutions of Poisson's equations. This alternative approach is called the analytical approach. Instead of the sequence of random walks, one considered a sequence of discrete Laplacians on the graphs and then proved that by choosing a proper scaling, those discrete Laplacians would converge to a good operator, called the Laplacian on the Sierpinski gasket. More precisely, let $\ell(V_m) = \{f : f \text{ maps } V_m \text{ to } \mathbb{R}\}$. Then define a linear operator $L_m : \ell(V_m) \to \ell(V_m)$ by

$$(L_m u)(p) = \sum_{q \in V_{m,p}} (u(q) - u(p))$$

for any $u \in \ell(V_m)$ and any $p \in V_m$. This operator L_m is the natural discrete Laplacian on the graph G_m. Then the Laplacian on the Sierpinski gasket, denoted by Δ, is defined by

$$5^m (L_m u)(p) \to (\Delta u)(p)$$

as $m \to \infty$. This Δ is now called the standard Laplacian on the Sierpinski gasket. (Of course, it needs to be shown that Δ is a meaningful operator in some sense with a non-trivial domain, as we will show in the course of this book. Also we will explain why 5^m is the proper scaling. See 3.7, in particular, Example 3.7.3.) This analytical approach was followed by Kusuoka [107] and Kigami [83], where they extended the construction of the Laplacians to more general class of finitely ramified fractals.

Since those early studies, many people have studied analysis on fractals and obtained numerous results using both approaches. Naturally the two approaches are complementary to each other and share the same goal. In this book, we will basically follow the analytical approach and study Dirichlet forms, Laplacians, eigenvalues of Laplacians and heat kernels on post critically finite self-similar sets. (Post critically finite self-similar sets are the mathematical formulation of finite ramified self-similar sets. See 1.3.) The advantage of the analytical approach is that one can get concrete and direct description of harmonic functions, Green's functions, Dirichlet forms and Laplacians. On the other hand, however, studying the detailed structure of the heat kernels, like the Aronson-type estimates, we need to employ the probabilistic approach. (Barlow's lecture note [6] is a self-contained and well-organized exposition in this direction. See also Kumagai [104] for a review of recent results.) Moreover, the probabilistic approach can be applied to infinitely ramified self-similar sets, for example, the Sierpinski carpet (Figure 0.4) as well. In the series of papers, [7, 8, 9, 10, 11, 12], Barlow and Bass constructed Brownian motions on the (higher dimensional) Sierpinski carpets and obtained the Aronson-type estimate of the associated heat kernels by using the probabilistic approach. Except for Kusuoka and Zhou [109], so far, the analytical approach has not succeeded in studying analysis on infinitely ramified fractals.

One may ask "why do you only study self-similar sets?". Indeed, self-similar sets are a special class of fractals and there are no objects in nature which have the exact structures of self-similar sets. The reason is that self-similar sets are perhaps the simplest and the most basic structures in the theory of fractals. They should give us much information on what would happen in the general case of fractals. Although there have been many studies on analysis on fractals, we are still near the beginning in the exploration of this field. We hope that this volume will contribute to fruitful developments in the future.

The organization of this book is as follows. In the first chapter, we will explain the basics of the geometry of self-similar sets. We will give the definition of self-similar sets, study topological structures of self-similar sets and introduce self-similar measures on them. The key notion is the self-

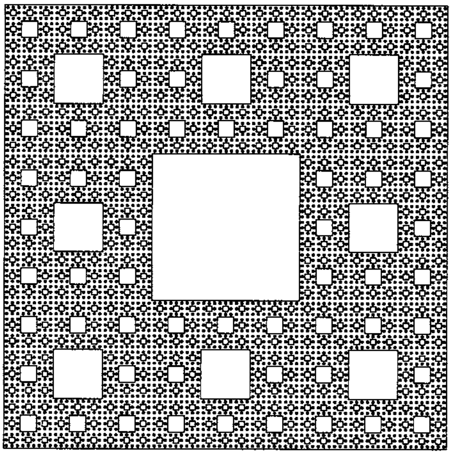

Fig. 0.4. Sierpinski carpet

similar structure which is a purely topological description of a self-similar set. See 1.3. Also, we will define post critically finite self-similar structure in 1.3, which will be our main stage of analysis on fractals.

In Chapter 2, we will study analysis on finite sets, namely, Dirichlet forms and Laplacians. The important fact is that those notions are closely related to electrical networks and that the effective resistance associated with them gives a distance on the finite set. Getting much help from this analogy with electrical networks, we will study the convergence of Dirichlet forms on a sequence of finite sets. This convergence theory will play an essential role in constructing Dirichlet forms and Laplacians on post critically finite self-similar sets in the next chapter.

Chapter 3 is the heart of this book, where we will explain how to construct Dirichlet forms, harmonic functions, Green's functions and Laplacians on post critically finite self-similar sets. The key notion here is the "harmonic structure" introduced in 3.1. In this chapter, we will spend many pages to argue how to deal with the case when a harmonic structure is not regular and also when $K\backslash V_0$ is not connected, where K is the self-similar set and V_0 corresponds to the boundary of K. These cases are of interest and sometimes really make a difference. However one would still get most of the essence of the theory by assuming that the harmonic structure is regular and that $K\backslash V_0$ is connected. So the reader may do so to avoid too many proofs.

In Chapter 4, we will study eigenvalues and eigenfunctions of Laplacians on post critically finite self-similar sets. We will obtain a Weyl-type estimate of the distribution of eigenvalues in 4.1 and show the existence of localized eigenfunctions in 4.4.

In the final chapter, we will study (Dirichlet or Neumann) heat kernels associated with Laplacians (or Dirichlet forms). In 5.2, it will be shown that the parabolic maximum principle holds for solutions of the heat equations. In 5.3, we will get on-diagonal estimates of heat kernels as time goes to zero.

This book is based on my graduate course at Cornell University in the fall semester, 1997. I would like to thank the Department of Mathematics, Cornell University for their hospitality. In particular, I would like to express my sincere gratitude to Professor R. S. Strichartz, who suggested that I wrote these lecture notes, and gave me many fruitful comments on the manuscript. I also thank Dr. C. Blum and Dr. A. Teplyaev who attended my lecture and gave me many useful suggestions. I am also grateful to the Isaac Newton Institute of Mathematical Science, University of Cambridge, where a considerable part of the manuscript was written during my stay. I would express my special thanks to Professors M. T. Barlow and R. F. Bass who carefully read the whole manuscript and helped me to improve my written English. I would also like to thank all the people who gave me valuable comments on the material; among them are Professors M. L. Lapidus, B. M. Hambly, V. Metz, T. Kumagai, Mr. T. Shimono and Mr. K. Kuwada. Finally I would like thank the late Professor Masaya Yamaguti, who was my thesis adviser and introduced me to the study of analysis on fractals.

1

Geometry of Self-Similar Sets

In this chapter, we will review some basics on the geometry of self-similar sets which will be needed later. Specifically, we will explain what a self-similar set is (in 1.1), how to understand the structure of a self-similar set (in 1.2 and 1.3) and how to calculate the Hausdorff dimension of a self-similar set (in 1.5).

The key notion is that of a "self-similar structure" introduced in 1.3, which is a description of a self-similar set from a purely topological viewpoint. As we will explain in 1.3, the topological structure of a self-similar set is essential in constructing analytical structures like Laplacians and Dirichlet forms. More precisely, if two self-similar sets are topologically the same (i.e., homeomorphic), then analytical structure on one self-similar set can be transferred to another self-similar set through the homeomorphism.

In particular, we will introduce the notion of post critically finite self-similar structures, on which we will construct the analytical structures like Laplacians and Dirichlet forms in Chapter 3.

1.1 Construction of self-similar sets

In this section, we will define self-similar sets on a metric space and show an existence and uniqueness theorem for self-similar sets. First we will introduce the notion of contractions on a metric space.

Notation. Let (X, d) be a metric space. For $x \in X$ and $r > 0$,

$$B_r(x) = \{y : y \in X, d(x, y) \leq r\}$$

Definition 1.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map f :

$X \to Y$ is said to be (uniformly) Lipschitz continuous on X with respect to d_X, d_Y if

$$L = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x,y)} < \infty.$$

The above constant L is called the Lipschitz constant of f and is denoted by $L = \text{Lip}(f)$.

Obviously, by the above definition, a Lipschitz continuous map is continuous.

Definition 1.1.2 (Contraction). Let (X, d) be a metric space. If $f : X \to X$ is Lipschitz continuous on X with respect to d and $\text{Lip}(f) < 1$, then f is called a contraction with respect to the metric d with contraction ratio $\text{Lip}(f)$. In particular, a contraction f with contraction ratio r is called a similitude if $d(f(x), f(y)) = rd(x,y)$ for all $x, y \in X$.

Remark. If f is a similitude on $(\mathbb{R}^n, d_E)$, where d_E is the Euclidean distance on $\mathbb{R}^n$, then there exist $a \in \mathbb{R}^n$, $U \in O(n)$ and $r > 0$ such that $f(x) = rUx + a$ for all $x \in \mathbb{R}^n$. (Exercise 1.1)

The following theorem is called the "contraction principle".

Theorem 1.1.3 (Contraction principle). *Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction with respect to the metric d. Then there exists a unique fixed point of f, in other words, there exists a unique solution to the equation $f(x) = x$. Moreover if x_* is the fixed point of f, then $\{f^n(a)\}_{n \geq 0}$ converges to x_* for all $a \in X$ where f^n is the n-th iteration of f.*

Proof. If r is the ratio of contraction of f, then for $m > n$,

$$\begin{aligned} d(f^n(a), f^m(a)) &\leq d(f^n(a), f^{n+1}(a)) + \cdots + d(f^{m-1}(a), f^m(a)) \\ &\leq (r^n + \cdots + r^{m-1}) d(a, f(a)) \leq \frac{r^n}{(1-r)} d(a, f(a)). \end{aligned}$$

Hence $\{f^n(a)\}_{n \geq 0}$ is a Cauchy sequence. As (X, d) is complete, there exists $x_* \in X$ such that $f^n(a) \to x_*$ as $n \to \infty$. Using the fact that $f^{(n+1)}(a) = f(f^n(a))$, we can easily deduce that $x_* = f(x_*)$.

Now, if $f(x) = x$ and $f(y) = y$, then $d(x,y) = d(f(x), f(y)) \leq rd(x,y)$. Therefore $d(x,y) = 0$ and $x = y$. So we have uniqueness of fixed points. □

Remark. In general, for a mapping f from a set to itself, a solution of $f(x) = x$ is called a fixed point or an equilibrium point of f.

We now state the main theorem of this section, which ensures uniqueness and existence of self-similar sets.

Theorem 1.1.4. *Let (X, d) be a complete metric space. If $f_i : X \to X$ is a contraction with respect to the metric d for $i = 1, 2, \ldots, N$, then there exists a unique non-empty compact subset K of X that satisfies*

$$K = f_1(K) \cup \cdots \cup f_N(K).$$

K is called the self-similar set with respect to $\{f_1, f_2, \ldots, f_N\}$.

Remark. In other literature, the name "self-similar set" is used in a more restricted sense. For example, Hutchinson [76] uses the name "self-similar set" only if all the contractions are similitudes. Also, in the case that all the contractions are affine functions on $\mathbb{R}^n$, the associated set may be called a self-affine set.

The contraction principle is a special case of Theorem 1.1.4 where $N = 1$. In the rest of this section, we will give a proof of Theorem 1.1.4. Define

$$F(A) = \bigcup_{1 \le j \le N} f_j(A)$$

for $A \subseteq X$. The main idea is to show existence of a fixed point of F. In order to do so, first we choose a good domain for F, defined by

$$\mathcal{C}(X) = \{A : A \text{ is a non-empty compact subset of } X\}.$$

Obviously F is a mapping from $\mathcal{C}(X)$ to itself. Next we define a metric δ on $\mathcal{C}(X)$, which is called the Hausdorff metric on $\mathcal{C}(X)$.

Proposition 1.1.5. *For $A, B \in \mathcal{C}(X)$, define*

$$\delta(A, B) = \inf\{r > 0 : U_r(A) \supseteq B \text{ and } U_r(B) \supseteq A\},$$

where $U_r(A) = \{x \in X : d(x, y) \le r$ for some $y \in A\} = \cup_{y \in A} B_r(y)$. Then δ is a metric on $\mathcal{C}(X)$. Moreover if (X, d) is complete, then $(\mathcal{C}(X), \delta)$ is also complete.

Before giving a proof of the above proposition, we recall some standard definitions in general topology.

Definition 1.1.6. Let (X, d) be a metric space and let K be a subset of X.

(1) A finite set $A \subset K$ is called an r-net of K for $r > 0$ if and only if $\cup_{x \in A} B_r(x) \supseteq K$.

(2) K is said to be totally bounded if and only if there exists an r-net of K for any $r > 0$.