IRRESISTIBLE INTEGRALS

The problem of evaluating integrals is well known to every student who has had a year of calculus. It was an especially important subject in nineteenth-century analysis and it has now been revived with the appearance of symbolic languages. In this book, the authors use the problem of exact evaluation of definite integrals as a starting point for exploring many areas of mathematics. The questions discussed here are as old as calculus itself.

In presenting the combination of methods required for the evaluation of most integrals, the authors take the most interesting, rather than the shortest, path to the results. Along the way, they illuminate connections with many subjects, including analysis, number theory, and algebra. This will be a guided tour of exciting discovery for undergraduates and their teachers in mathematics, computer science, physics, and engineering.

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IRRESISTIBLE INTEGRALS

Symbolics, Analysis and Experiments in the Evaluation of Integrals

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To Marian

To Lisa, Alexander and Stefan

Contents

Preface

The idea of writing a book on all the areas of mathematics that appear in the evaluation of integrals occurred to us when we found many beautiful results scattered throughout the literature.

The original idea was naive: inspired by the paper "Integrals: An Introduction to Analytic Number Theory" by Ilan Vardi (1988) we decided to write a text in which we would prove every formula in *Table of Integrals, Series, and Products* by I. S. Gradshteyn and I. M. Rhyzik (1994) and its precursor by Bierens de Haan (1867). It took a short time to realize that this task was monumental.

In order to keep the book to a reasonable page limit, we have decided to keep the material at a level accesible to a junior/senior undergraduate student. We assume that the reader has a good knowledge of one-variable calculus and that he/she has had a class in which there has been some exposure to a rigorous proof. At Tulane University this is done in Discrete Mathematics, where the method of mathematical induction and the ideas behind recurrences are discussed in some detail, and in Real Analysis, where the student is exposed to the basic material of calculus, now with rigorous proofs. It is our experience that most students majoring in mathematics will have a class in linear algebra, but not all (we fear, few) study complex analysis. Therefore we have kept the use of these subjects to a minimum. In particular we have made an effort *not* to use complex analysis.

The goal of the book is to present to the reader the many facets involved in the evaluation of definite integrals. At the end, we decided to emphasize the connection with number theory. It is an unfortunate fact of undergraduate, and to some extent graduate, education that students tend to see mathematics as comprising distinct parts. We have tried to connect the discrete (prime numbers, binomial coefficients) with the continuous (integrals, special functions). The reader will tell if we have succeeded.

Many of the evaluations presented in this book involve parameters. These had to be restricted in order to make the resulting integrals convergent. We have decided not to write down these restrictions.

The symbolic language Mathematica™ is used throughout the book. We do not assume that the reader has much experience with this language, so we incorporate the commands employed by the authors in the text. We hope that the reader will be able to reproduce what we write. It has been our experience that the best way to learn a symbolic language is to learn the commands as you need them to attack your problem of interest. It is like learning a real language. You do not need to be fluent in Spanish in order to order *empanadas*, but more is required if you want to understand *Don Quixote*. This book is mostly at the empanada level.

Symbolic languages (like Mathematica) are in a constant state of improvement, thus the statement *this cannot be evaluated symbolically* should always be complemented with the phrase *at the time of writing this text*.

We have tried to motivate the results presented here, even to the point of *wasting* time. It is certainly shorter to present mathematics as facts followed by a proof, but that takes all the fun out of it.

Once the target audience was chosen we decided to write first about the elementary functions that the student encounters in the beginning sequence of courses. This constitutes the first seven chapters of the book. The last part of the book discusses different families of integrals. We begin with the study of a rational integral, and there we find a connection with the expansion of the double square root. The reader will find here a glimpse into the magic world of Ramanujan. The next three chapters contain the normal integral, the Eulerian integrals gamma and beta, and Euler's constant. The book concludes with a short study on the integrals that can be evaluated in terms of the famous Riemann zeta function and an introduction to logarithmic integrals; we finish with our master formula: a device that has produced many interesting evaluations.

We hope that the reader will see that with a good calculus background it is possible to enter the world of integrals and to experience some of its flavor. The more experienced reader will certainly know shorter proofs than the ones we have provided here. The beginning student should be able to read all the material presented here, accepting some results as given. Simply take these topics as a list of *things to learn later in life*.

As stated above, the main goal of the book is to evaluate integrals. We have tried to use this as *a springboard for many unexpected investigations and discoveries in mathematics* (quoted from an earlier review of this manuscript). We have tried to explore the many ramifications involved with a specific evaluation. We would be happy to hear about new ones.

The question of integrating certain functions produces many reactions. On page 580 of M. Spivak's calculus book (1980) we find

> The impossibility of integrating certain functions in elementary terms is one of the most esoteric subjects in mathematics

and this should be compared with G. H. Hardy's famous remark

> I could never resist an integral

and R. Askey's comment[1]

> If things are nice there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do.

We have tried to keep these last two remarks in mind while writing.

The exercises are an essential part of the text. We have included alternative proofs and other connections with the material presented in the chapter. The level of the exercises is uneven, and we have provided hints for the ones we consider more difficult. The projects are exercises that we have not done in complete detail. We have provided some ideas on how to proceed, but for some of them we simply do not know where they will end nor how hard they could be. The author would like to hear from the reader on the solutions to these questions.

Finally the word *Experiments* in the subtitle requires an explanation. These are *computer experiments* in which the reader is required to guess a closed form expression for an analytic object (usually a definite integral) from enough data produced by a symbolic language. The final goal of the experiment is to provide a proof of the closed form. In turn, these proofs suggest new experiments.

The author would like to acknowledge many people who contributed to this book:

- First of all my special thanks to Dante Manna, who checked every formula in the book. He made sure that every $f_n^{(i+1)}$ was not a mistake for $f_{n-1}^{(i)}$. Naturally all the psosible errors are the author's responsibility.
- Bruce Berndt, Doron Zeilberger who always answered my emails.
- Michael Trott at Wolfram Research, Inc. who always answered my most trivial questions about the Mathematica language.
- Sage Briscoe, Frank Dang, Michael Joyce, Roopa Nalam, and Kirk Soodhalter worked on portions of the manuscript while they were undergraduates at Tulane.

[1] Quoted from a transparency by Doron Zeilberger.

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Frontmatter
More information

- The students of SIMU 2000: Jenny Alvarez, Miguel Amadis, Encarnacion Gutierrez, Emilia Huerta, Aida Navarro, Lianette Passapera, Christian Roldan, Leobardo Rosales, Miguel Rosario, Maria Torres, David Uminsky, and Yvette Uresti and the teaching assistants: Dagan Karp and Jean Carlos Cortissoz.
- The students of SIMU 2002: Benjamin Aleman, Danielle Brooker, Sage Briscoe, Aaron Cardona, Angela Gallegos, Danielle Heckman, Laura Jimenez, Luis Medina, Jose Miranda, Sandra Moncada, Maria Osorio, and Juan Carlos Trujillo and the teaching assistants: Chris Duncan and Dante Manna.
- The organizers of SIMU: Ivelisse Rubio and Herbert Medina.
- The participants of a 1999 summer course on a preliminary version of this material given at Universidad Santa Maria, Valparaiso, Chile.

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George Boros passed away during the final stages of this project. I have often expressed the professional influence he had on me, showing that integrals were interesting and fun. It is impossible to put in words what he meant as a person. *We miss him.*

— Victor Moll
New Orleans
January 2004

Notation

The notation used throughout the book is standard:

$\mathbb{N} = \{1, 2, 3, \dots\}$ are the natural numbers.
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$ are the integers.
$\mathbb{R}$ are the real numbers and $\mathbb{R}^+$ are the positive reals.
$\ln x$ is the natural logarithm.
$\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ and $\{x\}$ is the fractional part.
$n!$ is the factorial of $n \in \mathbb{N}$.
$\binom{n}{k}$ are the binomial coefficients.
C_m is the central binomial coefficients $\binom{2m}{m}$.
$(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the ascending factorial or Pochhammer symbol.