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Excerpt
More information

1

Factorials and Binomial Coefficients

1.1. Introduction

In this chapter we discuss several properties of factorials and binomial coefficients. These functions will often appear as results of evaluations of definite integrals.

Definition 1.1.1. A function $f : \mathbb{N} \to \mathbb{N}$ is said to satisfy a **recurrence** if the value $f(n)$ is determined by the values $\{f(1), f(2), \dots, f(n-1)\}$. The recurrence is of **order** k if $f(n)$ is determined by the values $\{f(n-1), f(n-2), \dots, f(n-k)\}$, where k is a fixed positive integer. The notation f_n is sometimes used for $f(n)$.

For example, the **Fibonacci numbers** F_n satisfy the second-order recurrence

$$F_n = F_{n-1} + F_{n-2}. \tag{1.1.1}$$

Therefore, in order to compute F_n, one needs to know only F_1 and F_2. In this case $F_1 = 1$ and $F_2 = 1$. These values are called the **initial conditions** of the recurrence. The Mathematica command

```
F[n_]:= If[n==0,1, If[n==1,1, F[n-1]+F[n-2]]]
```

gives the value of F_n. The modified command

```
F[n_]:= F[n]= If[n==0,1, If[n==1,1, F[n-1]+F[n-2]]]
```

saves the previously computed values, so at every step there is a single sum to perform.

Exercise 1.1.1. Compare the times that it takes to evaluate

$$F_{30} = 832040 \tag{1.1.2}$$

using both versions of the function F.

A recurrence can also be used to *define* a sequence of numbers. For instance

$$D_{n+1} = n(D_n + D_{n-1}), \; n \geq 2 \tag{1.1.3}$$

with $D_1 = 0, \; D_2 = 1$ defines the **derangement numbers**. See Rosen (2003) for properties of this interesting sequence.

We now give a recursive definition of the factorials.

Definition 1.1.2. The **factorial** of $n \in \mathbb{N}$ is defined by

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1. \tag{1.1.4}$$

A recursive definition is given by

$$\begin{aligned} 1! &= 1 \\ n! &= n \times (n-1)!. \end{aligned} \tag{1.1.5}$$

The first exercise shows that the recursive definition characterizes $n!$. This technique will be used throughout the book: in order to prove some identity, you check that both sides satisfy the same recursion and that the initial conditions match.

Exercise 1.1.2. Prove that the factorial is the unique solution of the recursion

$$x_n = n \times x_{n-1} \tag{1.1.6}$$

satisfying the initial condition $x_1 = 1$. **Hint**. Let $y_n = x_n/n!$ and use (1.1.5) to produce a trivial recurrence for y_n.

Exercise 1.1.3. Establish the formula

$$D_n = n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{1.1.7}$$

Hint. Check that the right-hand side satisfies the same recurrence as D_n and then check the initial conditions.

The first values of the sequence $n!$ are

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \tag{1.1.8}$$

and these grow very fast. For instance

50! = 30414093201713378043612608166064768844377641568960512000000000000

and 1000! has 2568 digits.

Mathematica 1.1.1. The Mathematica command for $n!$ is `Factorial[n]`. The reader should check the value 1000! stated above. The number of digits of an integer can be obtained with the Mathematica command `Length[IntegerDigits[n]]`.

The next exercise illustrates the fact that the extension of a function from $\mathbb{N}$ to $\mathbb{R}$ sometimes produces unexpected results.

Exercise 1.1.4. Use Mathematica to check that $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$.

The exercise is one of the instances in which the factorial is connected to π, the fundamental constant of trigonometry. Later we will see that the growth of $n!$ as $n \to \infty$ is related to e: the base of natural logarithms. These issues will be discussed in Chapters 5 and 6, respectively. To get a complete explanation for the appearance of π, the reader will have to wait until Chapter 10 where we introduce the **gamma function**.

1.2. Prime Numbers and the Factorization of $n!$

In this section we discuss the factorization of $n!$ into prime factors.

Definition 1.2.1. An integer $n \in \mathbb{N}$ is **prime** if its only divisors are 1 and itself.

The reader is refered to Hardy and Wright (1979) and Ribenboim (1989) for more information about prime numbers. In particular, Ribenboim's first chapter contains many proofs of the fact that there are infinitely many primes. Much more information about primes can be found at the site

`http://www.utm.edu/research/primes/`

The set of prime numbers can be used as building blocks for all integers. This is the content of the **Fundamental Theorem of Arithmetic** stated below.

Theorem 1.2.1. *Every positive integer can be written as a product of prime numbers. This factorization is unique up to the order of the prime factors.*

The proof of this result appears in every introductory book in number theory. For example, see Andrews (1994), page 26, for the standard argument.

Mathematica 1.2.1. The Mathematica command `FactorInteger[n]` gives the complete factorization of the integer n. For example `FactorInteger[1001]` gives the prime factorization $1001 = 7 \cdot 11 \cdot 13$. The concept of prime factorization can now be extended to rational numbers by allowing negative exponents. For example

$$\frac{1001}{1003} = 7 \cdot 11 \cdot 13 \cdot 17^{-1} \cdot 59^{-1}. \tag{1.2.1}$$

The efficient complete factorization of a large integer n is one of the basic questions in computational number theory. The reader should be careful with requesting such a factorization from a symbolic language like Mathematica: the amount of time required can become very large. A safeguard is the command

```
FactorInteger[n, FactorComplete -> False]
```

which computes the small factors of n and leaves a part unfactored. The reader will find in Bressoud and Wagon (2000) more information about these issues.

Definition 1.2.2. Let p be prime and $r \in \mathbb{Q}^+$. Then there are unique integers a, b, not divisible by p, and $m \in \mathbb{Z}$ such that

$$r = \frac{a}{b} \times p^m. \tag{1.2.2}$$

The p-**adic valuation** of r is defined by

$$\nu_p(r) = p^{-m}. \tag{1.2.3}$$

The integer m in (1.2.2) will be called the **exponent of** p **in** m and will be denoted by $\mu_p(r)$, that is,

$$\nu_p(r) = p^{-\mu_p(r)}. \tag{1.2.4}$$

Extra 1.2.1. The p-adic valuation of a rational number gives a new way of measuring its size. In this context, a number is small if it is divisible by a large power of p. This is the basic idea behind *p-adic Analysis*. Nice introductions to this topic can be found in Gouvea (1997) and Hardy and Wright (1979).

Exercise 1.2.1. Prove that the valuation ν_p satisfies

$$\nu_p(r_1 r_2) = \nu_p(r_1) \times \nu_p(r_2),$$
$$\nu_p(r_1/r_2) = \nu_p(r_1)/\nu_p(r_2),$$

and

$$\nu_p(r_1 + r_2) \leq \text{Max}\left(\nu_p(r_1), \nu_p(r_2)\right),$$

with equality unless $\nu_p(r_1) = \nu_p(r_2)$.

Extra 1.2.2. The p-adic numbers have many surprising properties. For instance, a series converges p-adically if and only if the general term converges to 0.

Definition 1.2.3. The **floor** of $x \in \mathbb{R}$, denoted by $\lfloor x \rfloor$, is the smallest integer less or equal than x. The Mathematica command is `Floor[x]`.

We now show that the factorization of $n!$ can be obtained *without* actually computing its value. This is useful considering that $n!$ grows very fast—for instance 10000! has 35660 digits.

Theorem 1.2.2. *Let p be prime and $n \in \mathbb{N}$. The exponent of p in $n!$ is given by*

$$\mu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor. \tag{1.2.5}$$

Proof. In the product defining $n!$ one can divide out every multiple of p, and there are $\lfloor n/p \rfloor$ such numbers. The remaining factor might still be divisible by p and there are $\lfloor n/p^2 \rfloor$ such terms. Now continue with higher powers of p. □

Note that the sum in (1.2.5) is finite, ending as soon as $p^k > n$. Also, this sum allows the fast factorization of $n!$. The next exercise illustrates how to do it.

Exercise 1.2.2. Count the number of divisions required to obtain

$$50! = 2^{47} \cdot 3^{22} \cdot 5^{12} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47,$$

using (1.2.5).

Exercise 1.2.3. Prove that every prime $p \leq n$ appears in the prime factorization of $n!$ and that every prime $p > n/2$ appears to the first power.

There are many expressions for the function $\mu_p(n)$. We present a proof of one due to Legendre (1830). The result depends on the expansion of an integer in base p. The next exercise describes how to obtain such expansion.

Exercise 1.2.4. Let $n, p \in \mathbb{N}$. Prove that there are integers $n_0, n_1, \ldots, n_r$ such that

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \tag{1.2.6}$$

where $0 \leq n_i < p$ for $0 \leq i \leq r$. **Hint**. Recall the division algorithm: given $a, b \in \mathbb{N}$ there are integers q, r, with $0 \leq r < b$ such that $a = qb + r$. To obtain the coefficients n_i first divide n by p.

Theorem 1.2.3. *The exponent of p in $n!$ is given by*

$$\mu_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{1.2.7}$$

where $s_p(n) = n_0 + n_1 + \cdots + n_r$ is the sum of the base-p digits of n. In particular,

$$\mu_2(n!) = n - s_2(n). \tag{1.2.8}$$

Proof. Write n in base p as in (1.2.6). Then

$$\begin{aligned}\mu_p(n!) &= \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \\ &= (n_1 + n_2 p + \cdots + n_r p^{r-1}) + (n_2 + n_3 p + \cdots + n_r p^{r-2}) \\ &\quad + \cdots + n_r,\end{aligned}$$

so that

$$\begin{aligned}\mu_p(n!) &= n_1 + n_2(1+p) + n_3(1+p+p^2) + \cdots + n_r(1+p+\cdots+p^{r-1}) \\ &= \frac{1}{p-1}\left(n_1(p-1) + n_2(p^2-1) + \cdots + n_r(p^r-1)\right) \\ &= \frac{n - s_p(n)}{p-1}.\end{aligned}$$

□

Corollary 1.2.1. *The exponent of p in $n!$ satisfies*

$$\mu_p(n!) \leq \frac{n-1}{p-1}, \tag{1.2.9}$$

with equality if and only if n is a power of p.

Mathematica 1.2.2. The command `IntegerDigits[n,p]` gives the list of numbers n_i in Exercise 1.2.4.

Exercise 1.2.5. Define

$$A_1(m) = (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1). \qquad (1.2.10)$$

Prove that, for any prime $p \neq 2$,

$$\mu_p(A_1(m)) \geq \mu_p(m!). \qquad (1.2.11)$$

Hint. Let $a_m = \prod_{k=1}^{m}(4k-1)$ and $b_m = \prod_{k=1}^{m}(4k+1)$ so that a_m is the product of the least m positive integers congruent to 1 modulo 4. Observe that for $p \geq 3$ prime and $k \in \mathbb{N}$, exactly one of the first p^k positive integers congruent to 3 modulo 4 is divisible by p^k and the same is true for integers congruent to 1 modulo 4. Conclude that $A_1(m)$ is divisible by the odd part of $m!$. For instance,

$$\frac{A_1(30)}{30!} = \frac{3599377626563574070183375 33}{2^{24}}. \qquad (1.2.12)$$

The products in (1.2.10) will be considered in detail in Section 10.9.

1.3. The Role of Symbolic Languages

In this section we discuss how to use Mathematica to conjecture general closed form formulas. A simple example will illustrate the point.

Exercise 1.2.3 shows that $n!$ is divisible by a large number of consecutive prime numbers. We now turn this information around to empirically suggest closed-form formulas. Assume that in the middle of a calculation we have obtained the numbers

$$\begin{aligned} x_1 &= 5356234211328000 \\ x_2 &= 102793666719744000 \\ x_3 &= 2074369080655872000 \\ x_4 &= 43913881247588352000 \\ x_5 &= 973160803270656000000, \end{aligned}$$

and one hopes that these numbers obey a simple rule. The goal is to obtain a function $x : \mathbb{N} \to \mathbb{N}$ that interpolates the given values, that is, $x(i) = x_i$ for $1 \leq i \leq 5$. Naturally this question admits more than one solution, and we will use Mathematica to find one. The prime factorization of the data is

$$\begin{aligned}
x_1 &= 2^{23} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \\
x_2 &= 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \\
x_3 &= 2^{18} \cdot 3^{12} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\
x_4 &= 2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19^3 \\
x_5 &= 2^{22} \cdot 3^8 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19
\end{aligned}$$

and a moment of reflection reveals that x_i contains all primes less than $i + 15$. This is also true for $(i + 15)!$, leading to the consideration of $y_i = x_i/(i + 15)!$. We find that

$$\begin{aligned}
y_1 &= 256 \\
y_2 &= 289 \\
y_3 &= 324 \\
y_4 &= 361 \\
y_5 &= 400,
\end{aligned}$$

so that $y_i = (i + 15)^2$. Thus $x_i = (i + 15)^2 \times (i + 15)!$ is one of the possible rules for x_i. This can be then tested against more data, and if the rule still holds, we have produced the conjecture

$$z_i = i^2 \times i!, \tag{1.3.1}$$

where $z_i = x_{i+15}$.

Definition 1.3.1. Given a sequence of numbers $\{a_k : k \in \mathbb{N}\}$, the function

$$T(x) = \sum_{k=0}^{\infty} a_k x^k \tag{1.3.2}$$

is the **generating function** of the sequence. If the sequence is finite, then we obtain a **generating polynomial**

$$T_n(x) = \sum_{k=0}^{n} a_k x^k. \tag{1.3.3}$$

The generating function is one of the forms in which the sequence $\{a_k : 0 \leq k \leq n\}$ can be incorporated into an analytic object. Usually this makes it easier to perform calculations with them. Mathematica *knows* a large number of polynomials, so if $\{a_k\}$ is part of a known family, then a symbolic search will produce an expression for T_n.

Exercise 1.3.1. Obtain a closed-form for the generating function of the Fibonacci numbers. **Hint**. Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$ be the generating function. Multiply the recurrence (1.1.1) by x^n and sum from $n = 1$ to ∞. In order to manipulate the resulting series observe that

$$\begin{aligned}\sum_{n=1}^{\infty} F_{n+1} x^n &= \sum_{n=2}^{\infty} F_n x^{n-1} \\ &= \frac{1}{x}\left(f(x) - F_0 - F_1 x\right).\end{aligned}$$

The answer is $f(x) = x/(1 - x - x^2)$. The Mathematica command to generate the first n terms of this is

```
list[n_]:= CoefficientList
[Normal[Series[ x/(1-x-x^{2}), {x,0,n-1}]],x]
```

For example, `list[10]` gives $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34\}$.

It is often the case that the answer is expressed in terms of more complicated functions. For example, Mathematica evaluates the polynomial

$$G_n(x) = \sum_{k=0}^{n} k! x^k \tag{1.3.4}$$

as

$$G_n(x) = -\frac{e^{-1/x}}{x}\left\{\Gamma(0, -\tfrac{1}{x}) + (-1)^n \Gamma(n+2)\Gamma(-1-n, -\tfrac{1}{x})\right\}, \tag{1.3.5}$$

where e^u is the usual **exponential function**,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \tag{1.3.6}$$

is the **gamma function**, and

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\, dt \tag{1.3.7}$$

is the **incomplete gamma function**. The exponential function will be discussed in Chapter 5, the gamma function in Chapter 10, and the study of $\Gamma(a, x)$ is postponed until Volume 2.

1.4. The Binomial Theorem

The goal of this section is to recall the binomial theorem and use it to find closed-form expressions for a class of sums involving binomial coefficients.

Definition 1.4.1. The **binomial coefficient** is

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}, \quad 0 \leq k \leq n. \tag{1.4.1}$$

Theorem 1.4.1. *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{1.4.2}$$

Proof. We use induction. The identity $(a+b)^n = (a+b) \times (a+b)^{n-1}$ and the induction hypothesis yield

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k} b^k + \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k-1} b^{k+1} \\ &= a^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] a^{n-k} b^k + b^n.\end{aligned}$$

The result now follows from the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \tag{1.4.3}$$

that admits a direct proof using (1.4.1). □

Exercise 1.4.1. Check the details.

Note 1.4.1. The binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.4.4}$$

shows that $(1+x)^n$ is the generating function of the binomial coefficients

$$\left\{ \binom{n}{k} : 0 \leq k \leq n \right\}.$$