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CHAPTER ONE

Basic Mathematical Background

The goal of this chapter is twofold: first, to provide the basic mathematical background needed to read the rest of this book, and second, to give the reader the basic background and motivation to learn more about the topics covered in this chapter by use of, for example, the referenced books and papers. This background is necessary to better prepare the reader to work in the area of partial differential equations (PDEs) applied to image processing and computer vision. Topics covered include differential geometry, PDEs, variational formulations, and numerical analysis. Extensive treatment on these topics can be found in the following books, which are considered essential for the shelves of everybody involved in this topic:

1. Guggengheimer's book on differential geometry [166]. This is one of the few simple-to-read books that covers affine differential geometry, Cartan moving frames, and basic Lie group theory. A very enjoyable book.
2. Spivak's "encyclopedia" on differential geometry [374]. Reading any of the comprehensive five volumes is a great pleasure. The first volume provides the basic mathematical background, and the second volume contains most of the basic differential geometry needed for the work described in this book. The very intuitive way Spivak writes makes this book a great source for learning the topic.
3. DoCarmo's book on differential geometry [56]. This is a very formal presentation of the topic, and one of the classics in the area.
4. Blaschke's book on affine differential geometry [39]. This is the source of basic affine differential geometry. A few other books have been published, but this is still very useful, and may be the most useful of all. Unfortunately, it is in German. A translation of some of the parts of the book appears in Ref. [53]. Be aware that this translation

contains a large number of errors. I suggest you check with the original every time you want to be sure about a certain formula.

5. Cartan's book on moving frames [61]. What can I say, this is a must. It is comprehensive coverage of the moving-frames theory, including the projective case, which is not covered in this book (projective differential geometry can be found in Ref. [412]). If you want to own this book, ask any French mathematician and he or she will point you to a place in Paris where you can buy it (and all the rest of the classical French literature). And if you want to learn about the recent developments in this theory, read the recent papers by Fels and Olver [140, 141].
6. Olver's books on Lie theory and differential invariants [281, 283]. A comprehensive coverage of the topic by one of the leaders in the field.
7. Many books have been written on PDEs. Basic concepts can be found in almost any book on applied mathematics. I strongly recommend the relatively new book by Evans [125] and the classic book by Protter and Weinberger for the maximum principle [321].
8. For numerical analysis, an almost infinite number of books have been published. Of special interest for the topics of this book are the books by Sod [371] and LeVeque [240]. As mentioned in the Introduction, Sethian's book [361] is also an excellent source of the basic numerical analysis needed to implement many of the equations discussed in this book. We are all expecting Osher's book as well, so keep an eye open, and, until then, check his papers at the website given in Ref. [290].
9. For applied mathematics in general and calculus of variations in particular, I strongly suggest looking at the classics, the books of Strang [375] and Courant and Hilbert [104].

1.1. Planar Differential Geometry

To start the mathematical background, basic concepts on planar differential geometry are presented. A planar curve, which can be considered as the boundary of a planar object, is given by a map from the real line into the real plane. More formally, a map $\mathcal{C}(p) : [a, b] \in \mathbb{R} \to \mathbb{R}^2$ defines a planar curve, where p parameterizes the curve. For each value $p_0 \in [a, b]$, we obtain a point $\mathcal{C}(p_0) = [x(p_0), y(p_0)]$ on the plane.

If $\mathcal{C}(a) = \mathcal{C}(b)$, the curve is said to be a closed curve. If there exists at least one pair of parameters $p_0 \neq p_1$, $p_0, p_1 \in (a, b)$, such that $\mathcal{C}(p_0) = \mathcal{C}(p_1)$, then the curve has a self-intersection. Otherwise, the curve is said to be a simple curve. Throughout this section, we will assume that the curve is at least two times differentiable.

Up to now, the parameter p has been arbitrary. Basically, p tells us the velocity at which the curve travels. This velocity if given by the tangent vector

$$\frac{\partial \mathcal{C}}{\partial p}.$$

We now search for a very particular parameterization, denoted as Euclidean arc length(s), such that this vector is always a unit vector,

$$\left\| \frac{\partial \mathcal{C}}{\partial p} \right\| = 1,$$

where $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ is the classical Euclidean length. In terms of the arc length, the curve is not defined anymore on the interval $[a, b]$ but on some interval $[0, L]$, where L is the (Euclidean) length of the curve. The arc length is unique (up to a constant), and is obtained by means of the relation

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}s} = \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}s},$$

which leads to

$$\frac{\mathrm{d}s}{\mathrm{d}p} = \left[\left(\frac{\mathrm{d}x}{\mathrm{d}p} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}p} \right)^2 \right]^{1/2}$$

We should note that throughout this book we consider only rectifiable curves. These are curves with a finite length between every two points. This is also equivalent to saying that the functions $x(p)$ and $y(p)$ have bounded variation.

From the definition of arc length, the (Euclidean) length of a curve between two points $\mathcal{C}(p_0)$ and $\mathcal{C}(p_1)$ is then given by

$$L(p_0, p_1) = \int_{p_0}^{p_1} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}p} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}p} \right)^2 \right]^{1/2} \mathrm{d}p = \int_{s(p_0)}^{s(p_1)} \mathrm{d}s.$$

Euclidean Curvature

The Euclidean arc length is one of the two basic concepts in planar differential geometry. The second one is that of curvature, which we proceed to define now.

The condition for the arc-length parameterization means that the inner product of the tangent $\mathcal{C}_s$ with itself is a constant, equal to one (throughout this book, subscripts indicate derivatives):

$$\langle \mathcal{C}_s, \mathcal{C}_s \rangle = 1.$$

Computing derivatives, we obtain

$$\langle \mathcal{C}_s, \mathcal{C}_{ss} \rangle = 0.$$

The first and the second derivatives, according to arc length, are then vectors perpendicular to each other. Ignoring for a moment the sign, we can define the Euclidean curvature κ as the absolute value of the normal vector $\mathcal{C}_{ss}$:

$$\kappa := \|\mathcal{C}_{ss}\|. \tag{1.1}$$

If $\vec{\mathcal{T}}$ and $\vec{\mathcal{N}}$ stand for the unit Euclidean tangent and the unit Euclidean normal, respectively ($\vec{\mathcal{T}} \perp \vec{\mathcal{N}}$), then (now κ has the sign back)

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}s} = \vec{\mathcal{T}},$$

$$\frac{\mathrm{d}^2\mathcal{C}}{\mathrm{d}s^2} = \kappa\vec{\mathcal{N}},$$

and from this we obtain the Frenet equations:

$$\frac{\mathrm{d}\vec{\mathcal{T}}}{\mathrm{d}s} = \kappa\vec{\mathcal{N}},$$

$$\frac{\mathrm{d}\vec{\mathcal{N}}}{\mathrm{d}s} = -\kappa\vec{\mathcal{T}}.$$

Many other definitions of curvature, all leading of course to the same concept, exist, and all of them can be derived from each other. For example, if θ stands for the angle between $\vec{\mathcal{T}}$ and the x axis, then

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

This is easy to show:

$$\begin{aligned}\frac{\mathrm{d}\vec{\mathcal{T}}}{\mathrm{d}s} &= \frac{\mathrm{d}(\cos\theta, \sin\theta)}{\mathrm{d}s}\\ &= \frac{\mathrm{d}\theta}{\mathrm{d}s}(-\sin\theta, \cos\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}s}\vec{\mathcal{N}},\end{aligned}$$

and the result follows from the Frenet equations.

The curvature $\kappa(s)$ at a given point $\mathcal{C}(s)$ is also the inverse of the radius of the disk that best fits the curve at $\mathcal{C}(s)$. Best fit means that the disk is tangent to the curve at $\mathcal{C}(s)$ (and therefore its center is on the normal direction). This is called the osculating circle.

A curve is not always given by an explicit representation of the form $\mathcal{C}(p)$. In many cases, as we will see later in this book, a curve is given in implicit form as the level set of a two-dimensional (2D) function $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$, that is,

$$\mathcal{C} \equiv \{(x, y) : u(x, y) = 0\}.$$

It is important then to be able to compute the curvature of $\mathcal{C}$ given in this form. It is possible to show that

$$\kappa = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{\left(u_x^2 + u_y^2\right)^{3/2}}.$$

Basically, this result can easily be obtained from the following simple facts:

1. The unit normal $\vec{\mathcal{N}}$ is perpendicular to the level sets, and

$$\vec{\mathcal{N}} = +(-)\frac{\nabla u}{\|\nabla u\|}, \tag{1.2}$$

   where the sign depends on the direction selected for $\vec{\mathcal{N}}$. This follows from the definition of the gradient vector

$$\nabla u := \frac{\partial u}{\partial x}\vec{x} + \frac{\partial u}{\partial y}\vec{y}.$$

   Of course, the tangent $\vec{\mathcal{T}}$ to the curve $\mathcal{C}$ is also tangent to the level sets.
2. If $\vec{\mathcal{N}} = (n_1, n_2)$, then $\kappa = \mathrm{d}n_1/\mathrm{d}x + \mathrm{d}n_2/\mathrm{d}y$.

Curve Representation by Means of Curvature

A curve is uniquely represented, up to a rotation and a translation, by the function $\kappa(s)$, that is, by its curvature as a function of the arc length. This is a very important property, which means that the curvature is invariant to Euclidean motions. In other words, two curves obtained from each other by a rotation and a translation have exactly the same curvature function $\kappa(s)$. Moreover, a curve $\mathcal{C} = (x, y)$ can be reconstructed from the curvature by the following equations:

$$x = x_0 + \cos\alpha \int_{\theta_0}^{\theta} \frac{\cos(\theta - \theta_0)}{\kappa(\theta)} \mathrm{d}\theta,$$

$$y = y_0 + \cos\beta \int_{\theta_0}^{\theta} \frac{\sin(\theta - \theta_0)}{\kappa(\theta)} \mathrm{d}\theta,$$

where the constants x_0, y_0, α, β, and θ_0 represent the fact that the reconstruction is unique up to a rotation and a translation.

Some Global Properties and the Evolute

A number of basic global facts related to the Euclidean curvature are now presented:

1. There are only two curves with constant curvature: straight lines (zero curvature) and circles (curvature equal to the inverse of the radius). The only closed curve with constant curvature is then the circle.
2. Vertices are the points at which the first derivative of the curvature vanishes. Every closed curve has at least four of these points (four-vertex theorem).
3. The total curvature of a closed curve is a multiple of 2π (exactly 2π in the case of a simple curve).
4. Isoperimetric inequality: Among all closed single curves of length (perimeter) L, the circle of radius $L/2\pi$ defines the one with the largest area.

As pointed out when defining the osculating circle, the curvature is the inverse of the radius of the osculating circle. The centers of these circles are called centers of curvature, and their loci define the Euclidean evolute of the curve:

$$\mathcal{E}_{\mathcal{C}}(s) := \mathcal{C}(s) + \frac{1}{\kappa(s)} \vec{\mathcal{N}}(s). \tag{1.3}$$

The basic geometric properties of the evolute, such as tangent, normal, arc length, and curvature, can be directly computed from those of the curve. The fact that the evolute of a closed curve is not a smooth curve it is of particular interest, as it is easy to show that the evolute has a cusp for every vertex of the curve.

1.2. Affine Differential Geometry

All the concepts presented in Section 1.1 are just Euclidean invariant, that is, invariant to rotations and translations. We now extend the concepts to the affine group. For the projective group see for example [226].

A general affine transformation in the plane ($\mathbb{R}^2$) is defined as

$$\tilde{\mathcal{X}} = A\mathcal{X} + B, \tag{1.4}$$

where $\mathcal{X} \in \mathbb{R}^2$ is a vector, $A \in \mathrm{GL}_2^+(\mathbb{R})$ (the group of invertible real 2×2 matrices with positive determinant) is the affine matrix, and $B \in \mathbf{R}^2$ is a translation vector. It is easy to show that transformations of the type of Eq. (1.4) form a real algebraic group $\mathcal{A}$, called the group of proper affine motions. We also consider the case in which we restrict $A \in \mathrm{SL}_2(\mathbb{R})$ (i.e., the determinant of A is 1), in which case Eq. (1.4) gives us the group of special affine motions, $\mathcal{A}_{\mathrm{sp}}$.

Before proceeding, let us briefly recall the notion of invariant. (For more detailed and rigorous discussions, see Refs. [51, 111, and 166] and Section 1.7 on Lie groups later in this chapter.) A quantity Q is called an invariant of a Lie group G if whenever Q transforms into $\tilde{Q}$ by any transformation G, we obtain $\tilde{Q} = \Psi Q$, where Ψ is a function of the transformation alone. If $\Psi = 1$ for all transformations in $\mathcal{G}$, Q is called an absolute invariant [111]. What we call invariant here is sometimes referred to in the literature as relative invariant. (We discuss more on Lie groups in Section 1.7.)

In the case of Euclidean motions (A in Eq. (1.4) being a rotation matrix), we have already seen that the Euclidean curvature κ of a given plane curve, as defined in Section 1.1, is a differential invariant of the transformation. In the case of general affine transformations, in order to keep the invariance property, a new definition of curvature is necessary. In this section, this affine curvature is presented [51, 53, 166, 374]. See also Refs. [39 and 53] for general properties of affine differential geometry.

Let $\mathcal{C}(p) : S^1 \to \mathbb{R}^2$ be a simple curve with curve parameter p (where S^1 denotes the unit circle). We assume throughout this section that all of our mappings are sufficiently smooth, so that all the relevant derivatives may be defined. A reparameterization of $\mathcal{C}(p)$ to a new parameter s can be performed such that

$$[\mathcal{C}_s, \mathcal{C}_{ss}] = 1, \tag{1.5}$$

where $[\mathcal{X}, \mathcal{Y}]$ stands for the determinant of the 2×2 matrix whose columns are given by the vectors $\mathcal{X},\ \mathcal{Y} \in \mathbb{R}^2$. This is also the area of the parallelogram defined by the vectors. This relation is invariant under special affine transformations, and the parameter s is called the affine arc length. (As commonly done in the literature, we use s for both the Euclidean and the affine arc lengths, and the meaning will be clear from the context.) Setting

$$g(p) := [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3}, \tag{1.6}$$

we find that the parameter s is explicitly given by

$$s(p) = \int_0^p g(\xi)\mathrm{d}\xi. \tag{1.7}$$

This is easily obtained by means of the relation

$$1 = [\mathcal{C}_s, \mathcal{C}_{ss}] = \left[\mathcal{C}_p \frac{\mathrm{d}p}{\mathrm{d}s}, \mathcal{C}_{pp}\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \mathcal{C}_p \frac{\mathrm{d}^2 p}{\mathrm{d}s^2}\right].$$

Note that in the above standard definitions, we have assumed (of course) that g (the affine metric) is different from zero at each point of the curve, i.e., the curve has no inflection points. This assumption will be made throughout this section unless explicitly stated otherwise. In particular, the convex curves we consider will be strictly convex, i.e., will have strictly positive (Euclidean) curvature. Fortunately, inflection points, that is, points with $\kappa = 0$, are affine invariant. Therefore limiting ourself to convex curves is not a major limitation for most image processing and computer vision problems.

It is easy to see that the following relations hold:

$$\mathrm{d}s = g\mathrm{d}p, \tag{1.8}$$

$$\vec{\mathbf{T}} := \mathcal{C}_s = \mathcal{C}_p \frac{\mathrm{d}p}{\mathrm{d}s}, \tag{1.9}$$

$$\vec{\mathbf{N}} := \mathcal{C}_{ss} = \mathcal{C}_{pp}\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + \mathcal{C}_p \frac{\mathrm{d}^2 p}{\mathrm{d}s^2}. \tag{1.10}$$

$\vec{\mathbf{T}}$ is called the affine tangent vector and $\vec{\mathbf{N}}$ is the affine normal vector. These formulas help to derive the relations between the Euclidean and the affine arc lengths, tangents, and normals. For example, considering v to be the Euclidean arc length, we have that

$$\mathrm{d}s = \kappa^{1/3}\mathrm{d}v, \tag{1.11}$$

where $\mathrm{d}s$ is still the affine arc length and $\mathrm{d}v$ is the Euclidean arc length:

$$\vec{\mathbf{T}} = \kappa^{-1/3}\vec{\mathcal{T}},$$
$$\vec{\mathbf{N}} = \kappa^{1/3}\vec{\mathcal{N}} + f(\kappa, \kappa_p)\vec{\mathcal{T}},$$

where f is a function of the first and second derivatives of the Euclidean curvature.

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Affine Curvature

We now follow the same procedure as in the Euclidean case in order to obtain the affine curvature. By differentiating Eq. (1.5) we obtain

$$[\mathcal{C}_s, \mathcal{C}_{sss}] = 0. \tag{1.12}$$

Hence the two vectors $\mathcal{C}_s$ and $\mathcal{C}_{sss}$ are linearly dependent, and so there exists μ such that

$$\mathcal{C}_{sss} + \mu \mathcal{C}_s = 0. \tag{1.13}$$

Equation (1.13) implies (just compare the corresponding areas and recall that $[\mathcal{C}_s, \mathcal{C}_{ss}] = 1$) that

$$\mu = [\mathcal{C}_{ss}, \mathcal{C}_{sss}], \tag{1.14}$$

and μ is called the affine curvature. The affine curvature is the simplest nontrivial differential affine invariant of the curve $\mathcal{C}$ [53]. Note that μ can also be computed as

$$\mu = [\mathcal{C}_{ssss}, \mathcal{C}_s]. \tag{1.15}$$

For the exact expression of μ as a function of the original parameter p, see Ref. [53].

As pointed out in Section 1.1, in the Euclidean case constant Euclidean curvature κ is obtained for only circular arcs and straight lines. Further, the Euclidean osculating figure of a curve $\mathcal{C}(p)$ at a given point is always the circle with radius $1/\kappa$ whose center lies on the normal at the given point. In the affine case, the conics (parabola, ellipse, and hyperbola) are the only curves with constant affine curvature μ ($\mu = 0$, $\mu > 0$, and $\mu < 0$, respectively). Therefore the ellipse is the only closed curve with constant affine curvature. The affine osculating conic of a curve C at a noninflexion point is a parabola, ellipse, or hyperbola, depending on whether the affine curvature μ is zero, positive, or negative, respectively. This conic has a triple-point contact with the curve $\mathcal{C}$ at that point (same point, tangent, and second derivative, or Euclidean curvature).

Affine Invariants

Assuming the group of special affine motions, we can easily prove the absolute invariance of some of the concepts introduced above when $\tilde{\mathcal{C}}$ is obtained from $\mathcal{C}$ by means of an affine transformation, that is, the affine arc length, tangent, normal, and curvature, as well as the area, are absolute invariants for the group of special affine motions. In general, for $A \in \mathrm{GL}_2^+(\mathbb{R})$, we obtain

$$\mathrm{d}\tilde{s} = [A]^{1/3}\mathrm{d}s,$$

$$\tilde{\mathcal{C}}_{\tilde{s}} = A[A]^{-1/3}\mathcal{C}_s,$$

$$\tilde{\mathcal{C}}_{\tilde{s}\tilde{s}} = A[A]^{-2/3}\mathcal{C}_{ss},$$

$$\tilde{\mu} = [A]^{-2/3}\mu,$$

$$\mathrm{area}(\tilde{\mathcal{C}}) = [A]\mathrm{area}(\mathcal{C}).$$

Thus the affine properties remain invariant (relative) but not absolute invariants. For an extended analysis about curvature like invariants, see Ref. [51].

Global Affine Differential Geometry

As in the Euclidean case, we now give a number of global properties regarding affine differential geometry:

1. There are at least six points with $\mu_s = 0$ (affine vertices) in a closed convex curve.
2. Define the affine perimeter of a closed curve as

$$L := \oint g \mathrm{d}p = \int \mathrm{d}s.$$

   Then, from all closed convex curves with a constant area, the ellipse, and only the ellipse, attains the greatest affine perimeter. In other words, for an oval (strictly convex closed curve) the following relation holds:

$$8\pi^2\mathrm{area} - L^3 \geq 0,$$

   and equality holds for only the ellipse.
3. For closed convex sets (ovals), the following affine isoperimetric inequality holds:

$$2\oint \mu \mathrm{d}s \leq \frac{L^2}{\mathrm{area}}. \tag{1.16}$$

   See [246, 247] for other inequalities and [147] for a related Euclidean result.
4. In the important case of the ellipse, the relation between the affine curvature and the area is given by

$$\mu = \left(\frac{\pi}{\mathrm{area}}\right)^{2/3},$$

   where area $= \pi r_1 r_2$ is the area, and r_1 and r_2 are the ellipse radii.