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London Mathematical Society Lecture Note Series. 276

Singularities of Plane Curves

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK http://www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011–4211, USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

First published 2000

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this book is available from the British Library

Library of Congress Cataloging-in-Publication Data

Casas-Alvero, E. (Eduardo), 1948-
Singularities of plane curves / Eduardo Casas-Alvero.
p. cm.
Included bibliographical references and index.
ISBN 0 521 78959 1 (pb)
1. Curves, Plane. 2. Singularities (Mathematics) I. Title.

QA565 .C37 2000
516.3'5–dc21 00-027671

ISBN 0 521 78959 1 paperback

To Mercè, Carles and Eduard

Contents

Preface

The singularities of algebraic and analytic varieties constitute an old and today very active field of research which combines techniques and viewpoints from different mathematical fields such as Geometry, Algebra, Topology and Function Theory. Doubtless, the oldest and best understood singularities are those of plane curves. Some curves with singular points appear in the work of the ancient Greek geometers, and the first contribution to a systematic study of plane curve singularities is due to Isaac Newton. Even if some relevant questions still remain open, nowadays, after the work of geometers such as Puiseux, Smith, Noether, Halphen, Enriques and Zariski, there is a well established theory for the analysis and classification of the singularities of plane curves. I have intended in this book to give a precise and detailed account of most of the main facts of this theory from a decidedly geometrical viewpoint which, I hope, will not completely hide its more algebraic or topological aspects. Infinitely near points have been taken as the cornerstone of the presentation: main description and classification of singularities are stated using infinitely near points, and other properties and invariants, such as characteristic exponents, Milnor number, discriminantal index, semigroups, polar quotients, etc., are related to them.

Infinitely near points are a nice and old idea for describing singularities. They appear in the work of M. Noether and their geometry was extensively developed by Enriques ([35], Book IV). Since then the geometry of infinitely near points has seldom appeared in the mathematical literature, maybe due to the rather obscure and non-intrinsic way the Italians used to introduce infinitely near points, or also to the fact that they do not have a straightforward translation in terms of rings and ideals. Besides the book by Enriques and Chisini, other references are the survey in the first chapter of Zariski's book on surfaces [92], chapter XI of the classical book on curves by Semple and Kneebone [73], a nice paper by Van der Waerden [83] and section 5 of Zariski's paper on saturation [93]. Fortunately infinitely near points may be introduced today in a very precise and clear way. I believe that they give a very appealing picture of how singularities of plane curves behave, and hope that this book will contribute to show it. Recent updating and development of Enriques' theory of infinitely near imposed singularities (virtual multiplicities) has lead to a better understanding of polar curves, linear systems and ideals of $\mathbb{C}\{x, y\}$ most of whose fruits, I think, are still to come. Furthermore, the use of infinitely near points and their properties, such as proximity, satellitism, etc., in the study of singularities in a wider context (varieties other than plane curves, foliations...) is just beginning and seems to be a very promising approach. This adds a further interest to the study of infinitely near points developed here.

Most of the results presented in this book have already appeared in recent or classical literature in a more or less explicit way, and so they may hardly be considered as essentially new. I have tried to precise and update the oldest ones and to organize all of them in a self-contained and comprehensive exposition centred on infinitely near points, which perhaps is new. Nevertheless, most of the proofs given here are new, either because preceding proofs do not fit in the context, or, as for many of the classical results, because no proof according to the modern standards was available. Among other results whose proofs I believe are new, let me quote in particular the Enriques theorem relating characteristic exponents and infinitely near points on an irreducible germ of curve (5.5.1), the theorem about the generators of the semigroup of a branch (5.8.2), whose proof uses the way they are defined from the branch rather than their particular values, a restricted version of the Lê-Ramanujam's μ-constant theorem (7.3.7), the determination of the E-sufficiency degree (7.5.1 and 7.6.1) and Zariski's theorem on factorization of complete ideals, which is proved using clusters of base points (8.4.13). If some of the results in the book may be qualified as new, they come from further development of Enriques' theory of virtual multiplicities and its application to linear systems. Among them, I would like to mention the determination of the (infinitely near) singular points of a curve from the base points of its jacobian system (8.6.4).

Singularities of plane curves is a rather elementary subject in the sense that it may be developed without using very complicated techniques from Algebra, Algebraic Geometry or Topology. I have tried to keep within this elementary frame making the book accessible to graduate students. Algebraic prerequisites are the most common facts on polynomials, series, commutative rings and ideals. Analytic prerequisites are reduced to some basic notions on complex analytic functions including implicit and inverse map theorems. Neither Weierstrass preparation and division theorems for two variables, nor the algebraic properties of the rings of convergent series are assumed as prerequisites, as they will be easily derived (for two variables) from Puiseux's theorem in chapter 1. Some knowledge of topics from algebraic geometry, as those usually contained in a first course on algebraic curves ([37], for instance), is maybe advisable even if not strictly needed. In this way I hope this book will be useful to both the graduate students, as a first step into the field of singularities, and the non-specialists that need to know the basic facts about singularities of plane curves.

Before giving a short description of the contents of the different chapters, maybe it is fair to say something on what is not to be found here. Throughout the book the base field is the complex one and the frame will be more analytic than algebraic: this is perhaps the best for an introductory book as it makes things easier and allows objects to stay close to the intuition (curves have points, topology is the classical one, series may be convergent, etc.), but obviously this does not cover the case of non-zero characteristic. Many results in the book are still true in the abstract case of an arbitrary algebraically closed base field, but not all of them: for instance Newton–Puiseux's algorithm and Puiseux's theorem in chapter 2 do not hold in positive characteristic. The book by Campillo [16] is advisable for a further reading covering the particular phenomena due to the positive characteristic. On the other hand, the reader will not find here a topological study of plane curve singularities. Fortunately, the topological side is maybe the best covered in the literature: the nice book by Milnor [60] is an obligate reading for anyone interested in singularities, and furthermore, among others, there are the books by Brieskorn and Knörrer [13], Eisenbud and Neumann [33] and Dimca [30], the latter being not specifically devoted to plane curves. Even within the more geometrical frame, the present book is far from being complete. In particular Zariski saturation theory [91] is missing. This is an interesting way of understanding the classification of plane curve singularities but unfortunately it is beyond the scope of this book, as it is an essentially non-planar approach.

Switching to what may be found in the book, chapter 0 is of introductory nature: basic facts that are needed in subsequent chapters are quickly presented there, mainly to set definitions and notations, as most of them should be familiar to the reader. Often proofs are not given, but the reader is referred to other sources.

Chapter 1 is mainly devoted to give a constructive proof of the Puiseux theorem on the roots of a power series in two variables. From it we obtain the main algebraic properties of the ring of convergent power series in two variables, including Weierstrass' theorems.

Chapter 2 presents the properties of germs of curve that follow from Puiseux's theorem, namely the decomposition of a germ into irreducible ones and the parameterization of an irreducible germ by means of its Puiseux series. We use such parameterizations to introduce the intersection multiplicity of germs of curve and to prove its main properties.

Chapter 3 contains the basic facts of the geometry of infinitely near points, the definition of equisingularity (the main notion of equivalence of singularities) and the description of infinitely near points and equisingularity classes by means of Enriques diagrams. Two sections are devoted to Northcott's neighbouring rings, which are an algebraic counterpart of the infinitely near points on a curve and allow us to introduce and compute the order δ of a singularity. A section about the Artin theorem for plane curves closes the chapter.

Chapter 4 deals with the conditions of asking curves to go through certain infinitely near points with given multiplicities (virtual multiplicities) and the families they give rise to. Certain inequalities (proximity inequalities) determine whether these conditions may be fulfilled. In case of the proximity inequalities being not satisfied, an effective algorithm, named unloading, gives the generic behaviour of the curves to which the conditions are imposed. Dual graphs and their relationship to Enriques diagrams are presented in section 4.4. The last two sections are devoted to the adjoint curves and their characterization by means of the conductor ideal, and to the $Af + B\varphi$ theorem of M. Noether, both for arbitrary singularities.

Chapter 5 relates the two main classical ways of describing and classifying singularities, namely the infinitely near points (M. Noether, Enriques) and the Puiseux series and their characteristic exponents (Smith, Halphen). The link is given by the Enriques theorem, which is proved in section 5.5. Further sections are devoted to semigroups associated with irreducible germs and to Abhyankar's approximate roots of polynomials.

Chapter 6 is devoted to polar germs, which are a very sharp tool for analyzing singularities. Their first properties lead to the Plücker formula and to the definition and computation of the Milnor number. Behaviour and equisingularity classes of polar germs are then considered and used for uncovering some properties of germs of curve that depend on their (local) isomorphism classes and not only on their equisingularity classes. Next, decompositions of the polar germs and polar invariants for both irreducible an non-irreducible germs of curve are presented.

Chapter 7 contains a local study of linear families of germs of curve and their base points including a sort of local Bertini theorem about non-existence of variable infinitely near multiple points and a restricted version (for linear families) of the μ-constant theorem. Properties of linear systems are then applied to sufficiency problems, that is, to the determination of the equisingularity class or the isomorphism class of a germ of curve by finitely many monomials of its local equation: in particular we present results of Samuel about algebraicity of analytic germs, as well as results of Teissier and Kuo-Lu determining equisingularity-sufficiency in terms of polar invariants.

Chapter 8 presents a geometric theory of valuations of the ring of convergent series $\mathbb{C}\{x, y\}$. Next, the relationship between complete ideals (those defined by valuative conditions) and the base points of their corresponding linear systems is shown: this relationship allows us to give an explicit proof of Zariski's theorem about unique factorization of complete ideals. Back to polar germs, the last two sections deal with aspects closely related to its base points and the linear structure of the jacobian system.

An appendix includes a couple of affine global results that are presented as applications of the local theory. They are the theorem of Abhyankar-Moh on immersions of the affine line in the affine plane and Jung's theorem about generators of the group of algebraic automorphisms of the affine plane.

Exercises are proposed at the end of each chapter. They include extensions of the theory already developed, as well as applications of local results to global algebraic geometry. Results proved in exercises are used in other exercises, but not in the text.

I am not able to quote all sources which in one way or another have influenced this book, so let me just quote two of them which I believe are the most important. The first one is the already quoted book by Enriques and Chisini *Teoria geometrica delle equazioni e delle funzioni algebriche* [35], my first reading on singularities and still a frequent one: in spite of many obscure proofs and some mistakes, this is a book full of ideas where I managed for the first time to understand infinitely near points not just as intermediate steps in a desingularization procedure, but as the points of an infinitesimal space where the local geometry of singularities may be displayed. The second one is the entire work of Zariski on singularities as it is the source of all modern work, at least on the more algebraic side. In particular, without his very precise critical and foundational work I would have not been able to understand and handle most of the Italian ideas. Other specific original sources of the most important results will be quoted within the text as far as I am aware of them, but since I am afraid that these quotations may be incomplete, I apologize in advance for the missing ones. I have tried to make a consistent and proper use of classical nomenclature, keeping classical conventions when possible and using the old names only for notions which really agree with the classical ones.

During the preparation of this book I have circulated preliminary versions to many colleagues and students, who made very valuable comments. I want to thank all of them, and in particular M. Alberich, A. Campillo, F. Delgado, J. Ma. Giral, O. Lavila, V. Navarro, A. Nobile, R. Peraire, J. Roé and G. Welters: their encouragement, help and suggestions largely improved the final version.

I wish also to thank the staff of Cambridge University Press for their kind attention and careful work, and the D.G.I.C.Y.T. of the Spanish Governement and the D.G.U. of the Catalan Governement for financial support.

Barcelona, May 2000.