Chapter 0

Preliminaries

This chapter is devoted to setting our general assumptions and conventions, to fixing notations and to recalling some basic notions and results in the form to be used throughout this book. The reader is assumed to be familiar with some very basic notions relating to analytic functions of several variables, such as germs of functions, varieties, manifolds, and analytic maps, including the inverse and implicit mapping theorems. For them he is referred to the first chapters of any book on analytic functions of several variables, such as for instance [41] or [40]. We will also make use of some rather elementary facts from ring and ideal theory: they can be found in general books such as [48] and, of course, also in those specifically devoted to commutative Algebra, [58], [8] or [32], for instance.

Throughout the book we will denote by $\mathbb{Z}$ the ring of integers, by $\mathbb{N}$ the set of the natural or positive integers and by $\mathbb{R}$ and $\mathbb{C}$ the fields of the real and complex numbers. We will use the symbol ∞ with the usual algebraic rules and the total order of $\mathbb{Z}$ extended so that $\infty \geq n$ for any $n \in \mathbb{Z}$. Domain means connected non-empty open set and, unless otherwise stated, all neighbourhoods of points will be assumed to be open and connected.

0.1 Projective spaces

A *projective space* of dimension d over a field K is a set $\mathbb{P}$ together with a exhaustive map $\pi : F - \{0\} \longrightarrow \mathbb{P}$, where F is a $(d+1)$-dimensional K-vector space and $\pi(v) = \pi(w)$ if and only if $v = aw$ for some $a \in K$. We often say that π defines a structure of projective space on $\mathbb{P}$. Once a basis in F is fixed the components of a non-zero vector v will be taken as homogeneous coordinates of $\pi(v)$, homogeneous coordinates of a point being thus determined up to a multiplicative constant. We will usually write $[x_0, \ldots, x_n]$ for the point of homogeneous coordinates $x_0, \ldots, x_n$. *Projectivities* (sometimes also called linear projectivities) are the maps between projective spaces induced by linear isomorphisms between the corresponding vector spaces, or, equivalently, invertible maps that are linear in homogeneous coordinates.

If $\mathbb{P}_1$ is a one-dimensional projective space and z_0, z_1 are projective coordinates on $\mathbb{P}_1$, the one to one map

$$\begin{aligned} \mathbb{P}_1 &\longrightarrow \mathbb{C} \cup \{\infty\} \\ [z_0, z_1] &\longmapsto z_0/z_1 \end{aligned}$$

which is assumed to send the point $[1,0]$ to ∞, will be called an *absolute coordinate* on $\mathbb{P}_1$.

0.2 Power series

We will denote, as customary, by $K[x_1, \ldots, x_n]$ and $K[[x_1, \ldots, x_n]]$, respectively, the rings of polynomials and formal power series in the variables $x_1, \ldots, x_n$ with coefficients in the ring K. In the case $K = \mathbb{C}$, we denote by $\mathbb{C}\{x_1, \ldots, x_n\}$ the ring of the convergent power series. If s is a power series we use the notation $o(s)$ for the *order* of s: if $s \neq 0$, $s = \sum_{i>0} s_i$ with each s_i a homogeneous polynomial of degree i, $o(s) = \min\{i | s_i \neq 0\}$ and $o(0) = \infty$. Recall that o satisfies the characteristic properties of valuations, namely

$$o(s+s') \geq \min\{o(s), o(s')\} \quad \text{and} \quad o(ss') = o(s) + o(s').$$

If $s \in K[[x_1, \ldots, x_n]]$, $n > 1$, we will write $o_{x_i}(s)$ for the order of s considered as a series in the single variable x_i whose coefficients are series in the remaining variables.

Assume that K is a field. Then a series s either in $K[[x_1, \ldots, x_n]]$ or in $\mathbb{C}\{x_1, \ldots, x_n\}$ is invertible if and only if $o(s) = 0$, that is, s has non-zero independent term. The non-invertible elements are thus those in the ideal $(x_1, \ldots, x_n)$ in either $K[[x_1, \ldots x_n]]$ or $\mathbb{C}\{x_1, \ldots, x_n\}$. Therefore, this ideal is the only maximal one and both rings are local rings.

If s is a series in a single variable, $s \in K[[x]]$, we write also $o_x(s)$ for $o(s)$. If K is a field it is clear that one may write $s = x^{o(s)}u$, where u is an invertible series. Furthermore, if $K = \mathbb{C}$ and s is convergent, then u is convergent too. It follows that both the rings $K[[x]]$ and $\mathbb{C}\{x\}$ are principal, their ideals being those generated by the powers of x.

We will use *substitution* of series into series in both the formal and convergent cases. If $f = f(x_1, \ldots, x_n)$ is a series in n variables $x_1, \ldots, x_n$ and $g_i = g_i(y_1, \ldots, y_m)$ are non-invertible series in m variables, $i = 1, \ldots, n$, there is a well determined series in the variables $y_1, \ldots, y_m$, currently denoted by $f(g_1, \ldots, g_n)$, which is obtained by substituting g_i for x_i in f. Substitution induces a morphism of K-algebras

$$\begin{aligned} K[[x_1, \ldots, x_n]] &\longrightarrow K[[y_1, \ldots, y_m]] \\ f &\longmapsto f(g_1, \ldots, g_n) \end{aligned}$$

which is an isomorphism if K is a field, $m = n$ and the degree-one parts of the g_i are linearly independent linear forms. (see [95], Vol. II, Ch. VII, for instance).

Notice that the last condition may be equivalently stated by saying that the jacobian determinant of the g_i is invertible.

If $K = \mathbb{C}$ and the series f and g_i, $i = 1, \dots, n$ are convergent, then they define holomorphic maps $\bar{f} : V \longrightarrow \mathbb{C}$, $p \mapsto f(p)$ and $\psi : U \longrightarrow \mathbb{C}^n$, $q \mapsto (g_1(q), \dots, g_n(p))$, where V and U are suitable neighbourhoods of the origin in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. After a suitable shrinking of U one may consider the composite map $\bar{f} \circ \psi$ which is a holomorphic function. Its series expansion at the origin is just $f(g_1, \dots, g_n)$ which, therefore, is convergent too. Thus, if the g_i are convergent, the above substitution morphism restricts to a morphism between the corresponding rings of convergent series: if convergent series are identified with germs of holomorphic functions, then this morphism is just the *pull-back morphism* ψ^*associated with ψ, namely $\psi^*(\bar{f}) = \bar{f} \circ \psi$. Using the inverse mapping theorem one may easily prove that the substitution morphism is an isomorphism if and only if $n = m$ and the jacobian determinant $\partial(g_1, ..., g_n)/\partial(y_1, ..., y_n)$ is invertible, just as in the formal case.

0.3 Surfaces, local coordinates

We are specially interested in irreducible and smooth analytic surfaces, since on they are lying the curves we will study in this book. Unless otherwise stated, the word *surface* will mean connected two-dimensional complex analytic manifold throughout the book. Note that a domain in a surface is itself a surface. Because of its own definition, surfaces are covered by open sets, each of which is actually analytically isomorphic to an open subset of $\mathbb{C}^2$. Then, for any fixed point O on a surface there is an open neighbourhood U and functions $x, y : U \longrightarrow \mathbb{C}$ that give an analytic isomorphism $p \longmapsto (x(p), y(p))$ onto an open subset U' of $\mathbb{C}^2$ and furthermore $x(O) = y(O) = 0$: a such pair of functions will be called a system of *local coordinates* (on S) at O. After choosing a system of local coordinates, x, y, at O, we will often identify each point $p \in U$ and its image $(x(p), y(p)) \in U'$ and so we will write just (x, y) for the point whose local coordinates are x, y. The functions holomorphic (or analytic) in a neighbourhood of O in S are obtained as the compositions $f(x, y)$ of the local coordinates and the functions f holomorphic in a neighbourhood of $(0, 0)$ in $\mathbb{C}^2$. Thus, locally at O one may thing of these functions just as functions of two variables x, y, holomorphic in a neighbourhood of the origin, and each of them is represented, in a neighbourhood of O, as the sum of an uniquely determined convergent power series in x, y. A second pair of holomorphic functions x', y', defined in an open neighbourhood of O and such that $x'(O) = y'(O) = 0$, is another system of local coordinates at O if and only if the jacobian determinant $\partial(x', y')/\partial(x, y)$ is not zero at O.

0.4 Morphisms

Holomorphic maps of surfaces will be also called *analytic morphisms* or just *morphisms* for short, and biholomorphic maps will be also called *analytic isomorphisms* or just *isomorphisms*. Saying that $\varphi : S' \longrightarrow S$ is a morphism means thus that for each $O' \in S$ there are local coordinates x', y' in a neighbourhood U' of O', local coordinates x, y in a neighbourhood U' of $O = \varphi(O')$, and functions f_1, f_2 analytic in U' such that $\varphi(U') \subset U$ and the restriction of φ to U' is given by the rule $(x', y') \mapsto (f_1(x', y'), f_2(x', y'))$. In these conditions we will say that φ is represented in U, or locally at O, by the equations

$$x = f_1(x', y'), \quad y = f_2(x', y').$$

The inverse map theorem asserts that the above morphism φ restricts to an isomorphism between suitable neighbourhoods of O' and O if and only if the jacobian determinant $\partial(f_1, f_2)/\partial(x', y')$ is non-zero at O. In such a case it is said that O' is a *non-critical point* of φ. The reader may notice that the set of non-critical points of φ is, by ts own definition, an open subset of S'. Its complement, the set of *critical points* of φ, is thus closed. We will see next that it is either nowhere dense or the whole of S'. Indeed, call X the interior of the set of critical points: it is open and so, S' being assumed to be connected, it will be enough to see that it is also closed. Take any $O' \in S'$ and assume that φ is represented in a neighbourhod U of O' by the equations $x = f_1(x', y')$, $y = f_2(x', y')$. If there is a point $q \in U \cap X$, then $\partial(f_1, f_2)/\partial(x', y')$, which is analytic in U, identically vanishes in a neighbourhood of q in U, and hence in the whole of U. Thus $O' \in X$ and X is closed. All morphisms we will consider thuroughout this book will have some non-critical point and thus a nowhere dense set of critical points.

If S is a surface, an *S-surface* or a *surface over S* will mean a pair (S', φ') where S' is a surface and φ' a morphism $\varphi' : S' \longrightarrow S$, its *structural morphism*. An *S-morphism* (or *morphism over S*) between S-surfaces $(S_1, \varphi_1), (S_2, \varphi_2)$ is just an ordinary analytic morphism between the surfaces $\varphi : S_1 \longrightarrow S_2$ that commutes with their structural morphisms: $\varphi_1 = \varphi_2 \circ \varphi$.

Assume that S and S' are surfaces and that $\varphi : U \longrightarrow U'$ is an analytic isomorphism between domains $U \subset S$, $U' \subset S'$ whose graph is closed in $S \times S'$. Then, by identifying in the disjoint union $S \amalg S'$ the points p and $\varphi(p)$, for $p \in U$, we get a connected Hausdorff topological space $S \amalg_\varphi S'$ which has a well determined structure of analytic surface for which the natural maps $S \longrightarrow S \amalg_\varphi S'$ and $S' \longrightarrow S \amalg_\varphi S'$ induce isomorphisms between S and S' and their respective images ([41], V.5). We will refer to this surface as the surface obtained by *patching together* S and S' along φ.

0.5 Local rings

Fix a point O in a surface S. The germs at O of the functions holomorphic in a neighbourhood of O describe a local ring which will be called the *local ring* *of S at O* or else the *local ring of O on S*. It will be denoted by $\mathcal{O}_{S,O}$, and also by $\mathcal{O}_O$, or just by $\mathcal{O}$ if S and O are clear from the context. Similarly, we denote by $\mathcal{M}_{S,O}$ (or $\mathcal{M}_O$, or just $\mathcal{M}$) the maximal ideal of $\mathcal{O}_{S,O}$. The elements of $\mathcal{M}_{S,O}$ are the germs of functions f for which $f(O) = 0$. The residual field $\mathcal{O}_{S,O}/\mathcal{M}_{S,O}$ is just the field of complex numbers $\mathbb{C}$, the germ of any function f being congruent modulo $\mathcal{M}_{S,O}$ with that of the constant function $f(O)$. If x, y are local coordinates at O, the representation of each holomorphic function near O by a convergent series leads to an isomorphism of local rings between the local ring of S at O and that of convergent power series in x, y, $\mathcal{O}_{S,O} \simeq \mathbb{C}\{x, y\}$, and in particular the germs of any pair of local coordinates at O are a pair of generators of $\mathcal{M}_{S,O}$. In the sequel, since no confusion can be made, we will use the same notations for the local coordinates and for their germs at O. Notice that it is clear from the identification $\mathcal{O}_{S,O} \simeq \mathbb{C}\{x, y\}$ that all powers of $\mathcal{M}_{S,O}$ have finite codimension as linear subspaces of $\mathcal{O}_{S,O}$. Main algebraic properties of the local rings $\mathcal{O}_{S,O}$ will be obtained in chapter 1, after proving Puiseux's theorem.

Assume that S and S' are surfaces, $\varphi : U' \longrightarrow U$ is a morphism from a neighbourhood U' of a point O' in S' into a neighbourhood U of O in S and $\varphi(O') = O$. By composing the representatives of the germs of functions with φ, one gets a *pull-back morphism* of local rings $\varphi^* = \varphi^*_{O,O'} : \mathcal{O}_{S,O} \longrightarrow \mathcal{O}_{S',O'}$. Notice that φ^* is a local morphism, that is, $\varphi^*(\mathcal{M}_{S,O}) \subset \mathcal{M}_{S',O'}$. Obviously φ^* depends only on the germ of φ at O': restrictions of φ to smaller neighbourhoods of O' induce the same pull-back morphism.

It is easy to see that all pull-back morphisms $\varphi^*_{O',O}$ are monomorphisms, but for the case of φ having no non-critical point. Indeed, assume that g is analytic in a neighbourhood V of O and has non-zero germ at O. This implies that $T = \{p \in V | g(p) = 0\}$ is a nowhere dense subset of V. If $g \circ \varphi$ identically vanishes in a neighbourhod V' of O', then $\varphi(V') \subset T$. This forces all points in V' to be critical and so (see section 0.4) φ has no non-critical points.

The identity map clearly induces the identity map between local rings, $Id^*_S = Id_{\mathcal{O}_{S,O}}$. If ψ is another morphism from a neighbourhood of a third point $O'' \in S''$ into U', $\psi(O'') = O'$, then $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. In particular, if O' is a non-critical point of φ, then φ^* is an isomorphism of local rings. The converse is also true, see 0.6 below.

0.6 Tangent and cotangent spaces

The complex vector space $\Omega_{S,O} = \mathcal{M}_{S,O}/\mathcal{M}^2_{S,O}$ is (or may be identified with) the cotangent space of S at O, differentiation maps the germ f to the class df of $f - f(0) \mod \mathcal{M}^2_{S,O}$. If x, y are local coordinates at O, then $\mathcal{M}_{S,O} = (x, y)$, and so the classes dx, dy of x, y in $\Omega_{S,O}$ (the differentials of x, y at O, in fact) are a basis of $\Omega_{S,O}$ over $\mathbb{C}$. For any $f \in \mathcal{O}_{S,O}$, $df = (\partial f/\partial x)_O dx + (\partial f/\partial y)_O dy$.

We denote by $T_{S,O}$ the tangent space to S at O. No matter how the tangent space is defined (one way is to take it as the dual of the cotangent space), $\Omega_{S,O}$ is its dual space, and hence the elements of $\Omega_{S,O}$ are linear forms on $T_{S,O}$.

The one-dimensional linear subspaces of $T_{S,O}$ will be called the *tangent lines* or *tangent directions* to S at O, each non-zero tangent vector determines a such tangent line and the whole of tangent lines to S at O is a one dimensional complex projective space, usually called the *pencil of tangent lines at* O. Notice that, since the spaces are of dimension two, also each non-zero form in the cotangent space determines a tangent line at O, namely its own kernel. Once a system of local coordinates x, y at O has been chosen, one usually takes basis dx, dy for $\Omega_{S,O}$ and its dual for $T_{S,O}$.

Let $\varphi : U' \longrightarrow U \subset S$ be, as above, a morphism between neighbourhoods of $O' \in S'$ and $O \in S$, $\varphi(O') = O$: we denote by $d\varphi$ and $\partial\varphi$ the associated morphisms $d\varphi : \Omega_{S,O} \longrightarrow \Omega_{S',O'}$ and $\partial\varphi : T_{S',O'} \longrightarrow T_{S,O}$. The first one is just that induced by φ^*, while the second one is its dual. If φ is represented locally at O' by the equations $x = f_1(x', y')$, $y = f_2(x', y')$, then the matrix of $\partial\varphi$ is the jacobian matrix of f_1, f_2 valued at O'. Thus O' is a non-critical point of φ if and only if $\partial\varphi$ (or $d\varphi$) is an isomorphism. This is the case if φ^* is an isomorphism of local rings, as claimed at the end of section 0.5 above.

We will also consider the graduate ring $\mathcal{G}_{S,O}$ of $\mathcal{O}_{S,O}$, namely

$$\mathcal{G}_{S,O} = \bigoplus_{i\geq 0} \mathcal{M}^i_{S,O}/\mathcal{M}^{i+1}_{S,O}.$$

It is a graduate ring and its pieces of degrees zero and one are $\mathbb{C}$ and $\Omega_{S,O}$ respectively. The graduate ring $\mathcal{G}_{S,O}$ is a polynomial ring in two variables as it may be easily seen by identifying $\mathcal{O}_{S,O}$ with the ring of convergent power series in local coordinates x, y: then it is clear that each homogeneous piece $\mathcal{M}^i_{S,O}/\mathcal{M}^{i+1}_{S,O}$ is freely generated over $\mathbb{C}$ by the classes of the monomials of degree i in x, y which in turn are the monomials of degree i in the classes dx, dy of x, y. It turns out, in particular, that the elements of $\mathcal{G}_{S,O}$ are the polynomial functions on the tangent space $T_{S,O}$. If f is a non-zero germ of holomorphic function at O, then one has $f \in \mathcal{M}^e - \mathcal{M}^{e+1}$ for some $e \geq 0$. One says then that e is the *order* of f and that the class $[f]$ of f in $\mathcal{M}^e/\mathcal{M}^{e+1}$ is the *initial form* or the *leading form* of f. Of course $[f][g] = [fg]$.

Using coordinates gives an easier (but not intrinsic) description of the graduate ring. Fix local coordinates x, y at O, defined in an open neighbourhood U of O, and use them to identify U with an open neighbourhood U' of the origin in $\mathbb{C}^2$. Then we take $\mathbb{C}^2$ as its own tangent space at the origin and identify the local coordinates x, y with their differentials dx, dy. We have thus $\mathcal{O}_{S,O} = \mathbb{C}\{x, y\}$ and $\mathcal{G}_{S,O} = \mathbb{C}[x, y]$. If s is a convergent series in x, y, say $s = \sum_{i\geq e} s_i$, each s_i being a homogeneous polynomial in x, y of degree i and $s_e \neq 0$, then the order and the initial forms of s are just its order and initial form as a series, namely e and s_e. Nevertheless, it should be noticed that the linear structure of $\mathbb{C}^2$ has an intrinsic meaning when $\mathbb{C}^2$ is identified with the tangent space, but the same linear structure has no intrinsic meaning if it is translated to U: for instance the linear character of, say, $x + y$ is intrinsic if $x + y$ is taken as an initial form (i.e., x, y are understood as forms on the tangent space) but it depends on the choice of the coordinates if $x + y$ is considered as a germ of function.

0.7 Curves

Our main interest in this book is the local study of analytic curves lying on a (smooth) surface: these curves are locally flat in the sense that they are lying on surfaces which are locally isomorphic to planes. Just for this, from the local viewpoint there is no difference between these curves and those lying on a true plane $\mathbb{C}^2$ and so our study is certainly not more general than that of the germs of curves at the origin of $\mathbb{C}^2$. Nevertheless, since blowing up will give rise to surfaces other than the plane, and to (global) curves on them, it will be easier to place ourselves in the just apparently more general frame of studying local properties of curves on smooth surfaces.

Even if they are not our main field of interest, we need to consider curves some of whose parts are counted with multiplicities. Because of this we take the curves (and later their germs) as defined by their systems of equations, rather than just as sets of points: an *analytic curve* (or just *curve*, for short) ξ is defined in a non-empty open subset U of a surface S by a system $(U_i, f_i)_{i\in I}$, where each U_i is a domain in U, $U = \bigcup_i U_i$, each f_i is a non-zero holomorphic function in U_i, and for each $i, j \in I$ for which $U_i \cap U_j \neq \emptyset$, there is a function $u_{i,j}$, holomorphic and with no zeros in $U_i \cap U_j$ such that $f_i = u_{i,j} f_j$ in $U_i \cap U_j$. We will call f_i an *equation* of ξ in U_i, or just a *local equation* of ξ. Two systems $(U_i, f_i)_{i\in I}$, $(V_j, g_j)_{j\in J}$ give rise to the same curve if and only if f_i/g_j is holomorphic and has no zeros in $U_i \cap V_j$ if $U_i \cap V_j \neq \emptyset$, for $(i, j) \in I \times J$. Notice that the definition includes all the (non-necessarily reduced) algebraic curves on S if S itself is algebraic. In particular we have the affine and projective plane algebraic curves if S is the affine or the projective plane over $\mathbb{C}$. Our interests being local, our curves will be mostly defined by a single equation in a single domain. Then if f is holomorphic and not identically zero in a domain U, we just say the curve $\xi : f = 0$ or the curve $f = 0$ to mean the curve ξ which has equation f in U. In particular, if x, y are local coordinates at O, the curves $y = 0$ and $x = 0$ will be called the *x-axis* and the *y-axis*, or the *first* and *second axis*, respectively.

A point p *lies on* (or belongs to) the curve ξ defined by $(U_i, f_i)_{i\in I}$ if and only if p is in one of the U_i and then $f_i(p) = 0$, the same condition being then obviously satisfied for any other index j for which $p \in U_j$. The condition clearly does not depend on the system $(U_i, f_i)_{i\in I}$ but only on the curve ξ. If p lies on ξ we will equivalently say that ξ goes through p and write $p \in \xi$. The set of points of ξ, also called the *locus* of ξ, will be denoted by $|\xi|$. It clearly is an analytic subvariety and a nowhere dense subset of U. We will say that a curve ξ is compact or connected if its locus $|\xi|$ is so.

Given two curves ξ and ζ defined in the same open set U in a surface S, it is not restrictive to assume that they are defined by systems $\{U_i, f_i\}_{i\in I}$ and $\{U_i, g_i\}_{i\in I}$ with the same family of domains $\{U_i\}_{i\in I}$. Then the curve defined by $\{U_i, f_i g_i\}_{i\in I}$ will be called the curve *composed of* ξ and ζ, or the *sum* of ξ and ζ, and denoted by $\xi + \zeta$. Such a curve does not depend on the equations of ξ and ζ but only on the curves themselves and, obviously, $|\xi + \zeta| = |\xi| \cup |\zeta|$. Addition of curves is clearly commutative and associative. If there is ζ' defined in the same open set U as ξ and ζ, and $\xi = \zeta + \zeta'$, then we say that ξ *contains*

ζ, or that ζ is a *component* of ξ. In such a case ζ' is determined by ξ and ζ and we often write $\zeta' = \xi - \zeta$: indeed, if ξ contains ζ, local equations of ζ divide local equations of ξ in a neighbourhood of each point of U and the quotients are local equations of ζ'.

Let ξ be a curve defined by $(U_i, f_i)_{i\in I}$ in a domain $U = \bigcup_i U_i$ in S. If U' is a domain in a surface S' and $\varphi : U' \longrightarrow U$ a morphism such that $\varphi(U') \not\subset |\xi|$, then the restrictions of the pull-backs $f_i \circ \varphi$ to the connected components of the $\varphi^{-1}(U_i)$ are the equations of a curve $\varphi^*(\xi)$ in $\varphi^{-1}(U)$ which is called the *pull-back* or the *inverse image* of ξ by φ. Indeed, if one of the f_i identically vanishes in a conected component of $\varphi^{-1}(U_i)$, then an easy argument using the conectness of U' shows that all $\varphi^*(f_i)$ identically vanish and therefore $\varphi(U') \subset |\xi|$. After this the remaining conditions for the restrictions of the $\varphi^*(f_i)$ to define a curve are obviously satisfied. In the case $\varphi(U') \subset |\xi|$, $\varphi^*(\xi)$ remains undefined. Note that if this occurs, then φ has no non-critical point. Once again the definition does not depend on the system of local equations and it is clear that a point $p' \in S'$ lies on $\varphi^*(\xi)$ if and only if $\varphi(p)$ lies on ξ. If ζ is another curve defined in U, $\varphi^*(\xi + \zeta)$ is defined and equals $\varphi^*(\xi) + \varphi^*(\zeta)$ provided both $\varphi^*(\xi)$ and $\varphi^*(\zeta)$ are defined. As for the pull-back of functions or germs of functions, $(\varphi \circ \psi)^*(\xi) = \psi^*(\varphi^*(\xi))$ if $\psi : U'' \longrightarrow U'$ is another morphism and $\psi(U'') \not\subset |\varphi^*(\xi)|$. If in particular φ is an isomorphism, then one may transform curves in both senses and we will just write $\varphi(\xi')$ for $(\varphi^{-1})^*(\xi')$, ξ' being any curve defined in U'.

0.8 Germs of curves

Curves are restricted to smaller open sets just by restricting their equations. Two curves that restrict to the same one in a suitable open neighbourhood of a closed set K are said to have the same germ at K. We will often use germs of curves at points instead of the curves themselves, as the germs carry all the local information on the curves: a *germ of curve* at a point O is an equivalence class of curves defined in some neighbourhood of O, modulo the equivalence relation of having the same restriction to an open neighbourhood of O. The point O is then called the origin of the germ. If ξ is a curve, we will write ξ_O to denote its germ at O. We will often say just germ instead of germ of curve if no confusion may result.

It is clear that functions giving the same germ at O define in a neighbourhood of O curves that have the same germ at O: then we take as *equations of a germ* of curve ξ at O the germs at O of all equations of all representatives of ξ. Each germ of curve is then determined by any of its equations. If f is a non-zero germ of holomorphic function at O we will write $\xi : f = 0$ or just $f = 0$ to denote the germ of curve ξ at O with equation f.

Assume that ξ and ζ are curves defined by equations f and g, respectively, in a certain open neighbourhood of O. By the definition of curve, the restrictions of ξ and ζ to some open neighbourhood of O agree if and only if f/g is holomorphic and has no zeros in a neighbourhood of O, that is, if and only if the germ of

f/g is an invertible element of $\mathcal{O}_{S,O}$. It turns out that if f and g are now germs of holomorphic functions at O, $f,g \neq 0$, they are equations of the same germ if and only if the germ f/g is an invertible element of $\mathcal{O}_{S,O}$. Thus we get a one to one correspondence between germs of curve at O and non-zero principal ideals of $\mathcal{O}_{S,O}$: a germ ξ corresponds to the ideal generated by any of its equations. In the sequel we will call this ideal the *ideal of* ξ.

Let $\xi : f = 0$ be a germ of curve at O and $I = (f)$ its ideal. We can associate with ξ (or with I) the germ of set at O represented by the set of points $|\xi'|$ of any representative ξ' of ξ. We call it the *locus of* ξ and denote it by $|\xi|$. It may be also written $|\xi| = \mathbf{V}(I) = \mathbf{V}(f)$, according to the customary notations for the loci of zeros of ideals.

Notice that we are considering in particular the *empty germ of curve*, which is the germ at O of any curve going not through O and corresponds to the ideal (1): since no confusion may result, we will denote it by the symbol of the empty set, $\emptyset$. Of course the locus of the empty germ is the germ of the empty set.

Addition of germs at O may be equivalently defined either by adding suitable representatives or by multiplying their equations or ideals: we write $\xi + \zeta$ for the *sum* of the germs ξ and ζ, which we also call the germ *composed of* ξ and ζ: it has equation fg if f and g are equations of ξ and ζ. As for curves, if $\xi = \zeta + \zeta'$, we still write $\zeta = \xi - \zeta'$ and we say that ζ is *contained* in, or is a *component* of, ξ.

A non-empty germ γ is said to be *irreducible* if and only if it cannot be obtained as the sum of two non-empty germs. This is clearly the same as saying that the equations of γ are irreducible elements of the ring $\mathcal{O}_{S,O}$ or that the ideal of γ is a prime ideal. Unique decomposition of a germ as a sum of irreducible ones will be obtained in 2.1.1, as an easy consequence of the factoriality of $\mathcal{O}_{S,O}$.

A curve γ defined in a neighbourhood of a point O is said to be *irreducible at* O if and only if its germ γ_O at O is irreducible.

Assume that O' is a point on a surface S' and that $\varphi : U' \longrightarrow U$ is a morphism between neighbourhoods of O' and O, $\varphi(O') = O$. If $\xi : f = 0$ is a germ of curve at O and $\varphi^*(f) \neq 0$, then $\varphi^*(f)$ is the equation of a germ of curve at O' which is called the *inverse image* or *pull-back* of ξ at O'. It will be denoted by $\varphi^*(\xi)$ (or $\varphi^*_{O'}(\xi)$ if reference to the point O' is needed). Obviously both the condition $\varphi^*(f) \neq 0$ and the germ $\varphi^*(\xi)$ are independent of the choice of the equation f. As the reader may easily check, $\varphi^*(\xi)$ is defined if and only if a representative ξ' of ξ has its pull-back $\varphi^*(\xi')$ defined, in which case all representatives of ξ in neighbourhoods of O contained in U have their pull-backs defined and representing $\varphi^*(\xi)$. Since φ^* is a local morphism, $\varphi^*(\xi) = \emptyset$ if and only if $\xi = \emptyset$. As for curves, $\varphi^*(\xi + \zeta)$ is defined and equals $\varphi^*(\xi) + \varphi^*(\zeta)$ if ζ is another germ at O and both $\varphi^*(\xi)$ and $\varphi^*(\zeta)$ are defined. Also $(\varphi \circ \psi)^*(\xi)$ is defined and equals $\psi^*(\varphi^*(\xi))$ if ψ is a morphism from a neighbourhood of a third point $O'' \in S''$ in a neighbourhood of O', $\psi(O'') = O'$ and both $\varphi^*(\xi)$ and $\psi^*(\varphi^*(\xi))$ are defined.

If φ is biholomorphic, then germs of curve can be transformed in both senses by taking direct or inverse images of their equations by the isomorphism of local rings $\varphi^* : \mathcal{O}_{S,O} \longrightarrow \mathcal{O}_{S',O'}$. In such a case the germs ξ and $\varphi^*(\xi)$ are called *isomorphic* or *analytically isomorphic*. An *analytic invariant* (of germs of curve) is any map defined on germs of curve that maps any two isomorphic germs to the same image. Usually these invariants are referred to by the image of a germ rather than by the map itself. For instance, the reader may easily see, after its definition in section 0.9 below, that the multiplicity of a germ (or the map that sends each germ to its multiplicity) is an analytic invariant.

0.9 Multiplicity and tangent cone

Assume that ξ is a germ of curve at O and f and f' are equations of ξ. Then (0.8), one has $f' = uf$ with u invertible and hence $[f'] = [u][f]$, where the initial form $[u]$ of u is a non-zero complex number, just because u is invertible. It follows in particular that all equations of ξ have the same order: it is called the *multiplicity of the germ* ξ, and sometimes also the *multiplicity of* O *on* ξ. We will denote it by $e(\xi)$, or even by $e_O(\xi)$ if an explicit reference to the origin O of ξ is needed. Equivalently the multiplicity of $\xi : f = 0$ may be defined by the relation $f \in \mathcal{M}^{e(\xi)} - \mathcal{M}^{e(\xi)+1}$.

It is clear that $e(\xi) = 0$ if and only if ξ is empty, and also that $e(\xi + \zeta) = e(\xi) + e(\zeta)$ for any two germs ξ and ζ at O (*additivity of multiplicity*). One way for getting germs of arbitrary multiplicity is to take them as composed of germs of multiplicity one. Nevertheless it is worth noting that the multiplicity of a germ is not merely a consequence of the number of germs it is composed of, as there are irreducible germs of arbitrary multiplicity (see example 0.11.5 below).

We have just seen above that different equations of a germ ξ have proportional initial forms: all these initial forms define thus the same algebraic cone in the tangent space $T_{S,O}$: it is called the *tangent cone* to ξ. If x, y are local coordinates at O and $f = \sum_{i+j\geq e} a_{i,j}x^iy^j$ is an equation of ξ, $e = e(\xi)$, then the tangent cone to ξ has equation $\sum_{i+j=e} a_{i,j}x^iy^j = 0$. Of course, a initial form being a homogeneous polynomial in two variables, it is a product of linear factors and the tangent cone is composed of lines. Each line composing the tangent cone to ξ is called a *tangent line* or a *principal tangent to* ξ. Two germs are said to be *tangent* if and only if they share a principal tangent. Notice that the multiplicity of the germ equals the order of its tangent cone, that is, the number of principal tangents provided each tangent is counted according to its multiplicity as a component of the tangent cone. Notice also that the tangent cone to $\xi + \zeta$ is composed of the tangent cones to the germs ξ and ζ. We will see in 2.2.6 that the tangent cone to an irreducible germ γ is composed of a single line (necessarily counted with multiplicity $e(\gamma)$).

The notions introduced above for a germ will be applied to its representatives with explicit mention of the point O: if ξ is now a curve, the *multiplicity*, the *tangent cone* and the *principal tangents* of ξ at O are those of its germ ξ_O. In particular we will write $e_O(\xi)$ for the multiplicity of ξ at O, $e_O(\xi) = e(\xi_O)$. Two curves are said to be *tangent* at O if their germs at O are so. If $e = e_O(\xi)$, it is often said that O is an e-fold point of the curve ξ.