Chapter 1

C^*-Algebra Theory

This chapter contains some basic facts about C^*-algebras that the reader is assumed to be (or become) familiar with. There are very few proofs given in this chapter, and the reader is referred to other sources, for example Murphy's book [29], for details.

1.1 C^*-algebras and *-homomorphisms

Definition 1.1.1. A *C^*-algebra* A is an algebra over $\mathbb{C}$ with a *norm* $a \mapsto \|a\|$ and an *involution* $a \mapsto a^*$, $a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leqslant \|a\|\,\|b\|$ and $\|a^*a\| = \|a\|^2$ for every a, b in A.

The axioms for a C^*-algebra A above imply that the involution is isometric, i.e., $\|a\| = \|a^*\|$ for every a in A.

A C^*-algebra A is called *unital* if it has a multiplicative identity, which will be denoted by 1 or 1_A. A **-homomorphism* $\varphi\colon A \to B$ between C^*-algebras A and B is a linear and multiplicative map which satisfies $\varphi(a^*) = \varphi(a)^*$ for all a in A. If A and B are unital and $\varphi(1_A) = 1_B$, then φ is called *unital* (or *unit preserving*). A C^*-algebra is said to be *separable* if it contains a countable dense subset.

1.1.2 Sub-C^*-algebras and sub-*-algebras. A non-empty subset B of a C^*-algebra A is called a *sub-*-algebra* of A if it is a *-algebra with the operations given on A, that is, if it is closed under the algebraic operations:

$$\begin{array}{lrl} \text{addition} & A \times A \to A, & (a,b) \mapsto a+b, \\ \text{multiplication} & A \times A \to A, & (a,b) \mapsto ab, \\ \text{adjoint} & A \to A, & a \mapsto a^*, \\ \text{scalar multiplication} & \mathbb{C} \times A \to A, & (\alpha,a) \mapsto \alpha a. \end{array}$$

A *sub-C^*-algebra* of A is a non-empty subset of A which is a C^*-algebra with respect to the operations given on A. Hence, a non-empty subset B of a C^*-algebra A is a sub-C^*-algebra if and only if it is norm-closed and closed under the four algebraic operations listed above.

The norm-closure of a sub-*-algebra of a C^*-algebra is a sub-C^*-algebra. This follows from the fact that the four algebraic operations above are continuous.

Let A be a C^*-algebra, and let F be a subset of A. The sub-C^*-algebra of A generated by F, denoted by $C^*(F)$, is the smallest sub-C^*-algebra of A that contains F. In other words, $C^*(F)$ is the intersection of all sub-C^*-algebras of A that contain F. The C^*-algebra $C^*(F)$ can be concretely described as follows. For each natural number n put

$$W_n = \{x_1 x_2 \cdots x_n : x_j \in F \cup F^*\},$$

where $F^* = \{x^* \mid x \in F\}$, and put $W = \bigcup_{n=1}^{\infty} W_n$. The set W is the set of all *words* in $F \cup F^*$, and W_n is the set of words of length n. Using that $W = W^*$ and that W is closed under multiplication, we see that the linear span of W is a sub-*-algebra of A. It follows that

$$C^*(F) = \overline{\operatorname{span} W}.$$

We write $C^*(a_1, a_2, \ldots, a_n)$ instead of $C^*(\{a_1, a_2, \ldots, a_n\})$, when $a_1, a_2, \ldots, a_n$ are elements in A.

Theorem 1.1.3 (Gelfand–Naimark). *For each C^*-algebra A there exist a Hilbert space H and an isometric *-homomorphism φ from A into $B(H)$, the algebra of all bounded linear operators on H. In other words, every C^*-algebra is isomorphic to a sub-C^*-algebra of $B(H)$. If A is separable, then H can be chosen to be a separable Hilbert space.*

A proof can be found in [29, Theorem 3.4.1]. The Hilbert space H is obtained by viewing A as a vector space, equipping it with a suitable inner product (coming from a positive, linear functional on A), and forming the completion of A with respect to this inner product.

1.1.4 Ideals and quotients. By an ideal in a C^*-algebra we shall always understand a closed, two-sided ideal (unless otherwise stated). Every ideal is automatically self-adjoint, and thereby a sub-C^*-algebra (see [29, 3.1.3]).

Assume that I is an ideal in a C^*-algebra A. The quotient of A by I is

$$A/I = \{a + I : a \in A\}, \quad \|a + I\| = \inf\{\|a + x\| : x \in I\}, \quad \pi(a) = a + I.$$

In this way A/I becomes a C^*-algebra, $\pi\colon A \to A/I$ is a *-homomorphism, called the *quotient mapping*, and $I = \mathrm{Ker}(\pi)$ (see [29, 3.1.4]).

Let $\varphi\colon A \to B$ be a *-homomorphism. Then, automatically, $\|\varphi(a)\| \leqslant \|a\|$ for all a in A, and φ is injective if and only if φ is isometric (see Exercise 1.8 or [29, 3.1.5]). The kernel, $\mathrm{Ker}(\varphi)$, of φ is an ideal in A, and the image, $\mathrm{Im}(\varphi) = \varphi(A)$, of φ is a sub-C^*-algebra of B (see [29, 3.1.6]). By the first isomorphism theorem there is one (and only one) *-homomorphism $\varphi_0\colon A/\mathrm{Ker}(\varphi) \to B$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & B \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\varphi_0} & \\ & A/\mathrm{Ker}(\varphi) & \end{array}$$

commutes, i.e., $\varphi_0 \circ \pi = \varphi$. Moreover, φ_0 is injective.

A C^*-algebra A is called *simple* if the only ideals in A are the two trivial ideals 0 and A.

1.1.5 Short exact sequences. A (finite or infinite) sequence of C^*-algebras and *-homomorphisms

$$\cdots \longrightarrow A_n \xrightarrow{\ \varphi_n\ } A_{n+1} \xrightarrow{\ \varphi_{n+1}\ } A_{n+2} \longrightarrow \cdots$$

is said to be *exact* if $\mathrm{Im}(\varphi_n) = \mathrm{Ker}(\varphi_{n+1})$ for all n. An exact sequence of the form

$$0 \longrightarrow I \xrightarrow{\ \varphi\ } A \xrightarrow{\ \psi\ } B \longrightarrow 0 \tag{1.1}$$

is called *short exact*.

If I is an ideal in A, then

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

is a short exact sequence, where ι is the inclusion mapping and π is the quotient mapping. Conversely, given (1.1), then $\varphi(I)$ is an ideal in A, the C^*-algebra B is isomorphic to $A/\varphi(I)$, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\varphi} & A & \xrightarrow{\psi} & B & \longrightarrow & 0 \\ & & \varphi \downarrow \cong & & \| & & \downarrow \cong & & \\ 0 & \longrightarrow & \varphi(I) & \underset{\iota}{\hookrightarrow} & A & \underset{\pi}{\longrightarrow} & A/\varphi(I) & \longrightarrow & 0. \end{array}$$

If in (1.1) there is a *-homomorphism $\lambda\colon B \to A$ such that $\psi \circ \lambda = \mathrm{id}_B$, then λ is called a *lift* of ψ and (1.1) is said to be *split exact*. Not all short exact sequences are split exact, see Exercise 1.2.

The *direct sum* $A \oplus B$ of two C^*-algebras A and B is the C^*-algebra of all pairs (a, b), with $a \in A$ and $b \in B$, equipped with entry-wise defined algebraic operations and the norm

$$\|(a, b)\| = \max\{\|a\|, \|b\|\}.$$

Define $\iota_A\colon A \to A \oplus B$, $\iota_B\colon B \to A \oplus B$, $\pi_A\colon A \oplus B \to A$, and $\pi_B\colon A \oplus B \to B$ by $\iota_A(a) = (a, 0)$, $\iota_B(b) = (0, b)$, $\pi_A(a, b) = a$, and $\pi_B(a, b) = b$. Then

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \underset{\iota_B}{\overset{\pi_B}{\rightleftarrows}} B \longrightarrow 0$$

is a split exact sequence (with lift ι_B). Not all split exact sequences are direct sums, see Exercise 1.3 (vi) and Exercise 1.1.

1.1.6 Adjoining a unit. To every C^*-algebra A one can associate a unique unital C^*-algebra $\widetilde{A}$ that contains A as an ideal and with the property that $\widetilde{A}/A$ is isomorphic to $\mathbb{C}$. See Exercise 1.3 and [29, 2.1.6] for details.

Let $\pi\colon \widetilde{A} \to \mathbb{C}$ be the quotient mapping, and let $\lambda\colon \mathbb{C} \to \widetilde{A}$ be defined by $\lambda(\alpha) = \alpha 1_{\widetilde{A}}$. Then

$$0 \longrightarrow A \xrightarrow{\iota} \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \qquad (1.2)$$

is a split exact sequence, and

$$\widetilde{A} = \{a + \alpha 1_{\widetilde{A}} : \ a \in A, \ \alpha \in \mathbb{C}\}.$$

The C^*-algebra $\widetilde{A}$ is called the *unitization* of A, or A with a unit adjoined.

Adjoining a unit is functorial in the sense that if $\varphi \colon A \to B$ is a *-homomorphism, then there is one and only one *-homomorphism $\widetilde{\varphi} \colon \widetilde{A} \to \widetilde{B}$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \widetilde{A} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\widetilde{\varphi}}\downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & \widetilde{B} & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \end{array}$$

commutative. The *-homomorphism $\widetilde{\varphi}$ is given by $\widetilde{\varphi}(a + \alpha 1_{\widetilde{A}}) = \varphi(a) + \alpha 1_{\widetilde{B}}$, where a belongs to A and α is a complex number. Observe that $\widetilde{\varphi}$ is unit preserving.

If A is contained in a unital C^*-algebra B whose unit 1_B does not belong to A, then $\widetilde{A}$ is equal (or isomorphic) to the sub-C^*-algebra $A + \mathbb{C} \cdot 1_B$ of B. (See Exercise 1.3.)

If A has a unit 1_A, and if $1_{\widetilde{A}}$ as above is the unit in $\widetilde{A}$, then $f = 1_{\widetilde{A}} - 1_A$ is a projection in $\widetilde{A}$, and

$$\widetilde{A} = \{a + \alpha f : a \in A, \ \alpha \in \mathbb{C}\}.$$

The map $A \oplus \mathbb{C} \to \widetilde{A}$ given by $(a, \alpha) \mapsto a + \alpha f$ is a *-isomorphism, and $\widetilde{A}$ is isomorphic to $A \oplus \mathbb{C}$. If A is not unital, then $\widetilde{A}$ is not isomorphic to $A \oplus \mathbb{C}$ because $\widetilde{A}$ is unital and $A \oplus \mathbb{C}$ is not.

1.2 Spectral theory

1.2.1 The spectrum. Let A be a unital C^*-algebra and let a be an element in A. The *spectrum* of a is the set of complex numbers λ such that $a - \lambda \cdot 1$ is not invertible, and it is denoted by $\mathrm{sp}(a)$. The *spectral radius*, $r(a)$, of a is

$$r(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\}.$$

The spectrum $\mathrm{sp}(a)$ is a closed subset of $\mathbb{C}$, and the spectral radius satisfies $r(a) \leqslant \|a\|$ (see [29, Lemma 1.2.4] for these two facts). It follows that $\mathrm{sp}(a)$ is a compact subset of the complex plane. A more refined argument, using complex analysis, shows that the spectrum is non-empty, the sequence

$\{\|a^n\|^{1/n}\}_{n=1}^{\infty}$ is decreasing, and

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}. \tag{1.3}$$

If A is not unital, then embed A in its unitization $\widetilde{A}$ (see Paragraph 1.1.6) and let $\mathrm{sp}(a)$ be the spectrum of a viewed as an element in $\widetilde{A}$. If A is non-unital, then $0 \in \mathrm{sp}(a)$ for all a in A.

An element a in A is called

- *self-adjoint* if $a = a^*$,
- *normal* if $aa^* = a^*a$,
- *positive* if a is normal and $\mathrm{sp}(a) \subseteq \mathbb{R}^+$ (with the convention that 0 belongs to $\mathbb{R}^+$).

The set of positive elements in A is denoted by A^+. It is an elementary fact that every self-adjoint element has spectrum contained in $\mathbb{R}$. It is a much deeper fact that an element a in A is positive if and only if $a = x^*x$ for some x in A. For a normal element a, equation (1.3) reduces to $r(a) = \|a\|$.

The spectrum of an element a in a C^*-algebra A depends, a priori, on the ambient C^*-algebra A. However, if A is a *unital* C^*-algebra, B is a sub-C^*-algebra of A that contains the unit of A, and a is an element in B, then the spectrum of a relative to B is equal to the spectrum of a relative to A. When A is non-unital, or when A is unital, but the unit of A does not belong to B, then the spectrum, relative to A, of an element a in B consists of 0 and of all numbers in the spectrum of a relative to B. These claims follow from the fact that if a is an invertible element in a unital C^*-algebra A, then the inverse of a belongs to $C^*(a)$, the smallest sub-C^*-algebra of A containing a, see Exercise 1.6.

1.2.2 States. A linear map $\rho\colon A \to \mathbb{C}$ is called a (linear) functional. Let

$$\|\rho\| = \sup\{|\rho(a)| : a \in A,\ \|a\| \leqslant 1\}$$

denote its operator norm. A linear functional ρ is continuous if and only if $\|\rho\| < \infty$. If $\rho(a) \geqslant 0$ for every positive element a in A, then ρ is said to be positive. A *state* ρ on a *unital* C^*-algebra A is a positive linear functional with $\rho(1) = 1$, or, equivalently, with $\|\rho\| = 1$. The set of states on a unital C^*-algebra separates points in the sense that if a is an element in A such that $\rho(a) = 0$ for every state ρ on A, then $a = 0$.

The next theorem (and the additions to it) characterizing commutative C^*-algebras has the interpretation that a general (non-commutative) C^*-algebra can be viewed as a "non-commutative topological space", and that the study of C^*-algebras is the study of "non-commutative topology". K-theory for C^*-algebras is a branch of operator algebras where this analogy holds particularly well.

Theorem 1.2.3 (Gelfand). *Every Abelian C^*-algebra is isometrically *-isomorphic to the C^*-algebra $C_0(X)$ for some locally compact Hausdorff space X.*

Recall that $C_0(X)$ is the C^*-algebra of all continuous functions $f\colon X \to \mathbb{C}$ "vanishing at infinity": for each $\varepsilon > 0$ there is a compact subset K of X such that $|f(x)| \leqslant \varepsilon$ for all x in $X \setminus K$. The norm on $C_0(X)$ is the supremum norm. If X is compact, then $C_0(X)$ is equal to $C(X)$, the set of all continuous functions $f\colon X \to \mathbb{C}$.

In addition to Gelfand's theorem we have the following.

(i) $C_0(X)$ is unital if and only if X is compact.

(ii) $C_0(X)$ is separable if and only if X is separable.

(iii) X and Y are homeomorphic if and only if $C_0(X)$ and $C_0(Y)$ are isomorphic.

(iv) Each continuous function $g\colon Y \to X$ induces a *-homomorphism $\varphi : C_0(X) \to C_0(Y)$ by $\varphi(f) = f \circ g$, and, conversely, for every *-homomorphism $\varphi\colon C_0(X) \to C_0(Y)$ there is a continuous function $\widehat{\varphi}\colon Y \to X$, such that $\varphi(f) = f \circ \widehat{\varphi}$.

(v) There is a bijective correspondence between open subsets of X and ideals in $C_0(X)$. The ideal corresponding to the open subset U of X is the set of all f in $C_0(X)$ that vanish on the complement, U^c, of U, and this ideal is isomorphic to $C_0(U)$. (The isomorphism is given by restriction, and one shows that this map is surjective by invoking the Stone–Weierstrass theorem.) The *-homomorphism $C_0(X) \to C_0(U^c)$ given by restriction $f \mapsto f|_{U^c}$ is surjective (by the Stone–Weierstrass theorem), and so we get a short exact sequence:

$$0 \longrightarrow C_0(U) \longrightarrow C_0(X) \longrightarrow C_0(U^c) \longrightarrow 0.$$

The map $C_0(U) \to C_0(X)$ is given by extending a function f in $C_0(U)$ to X by giving it the value 0 on U^c.

1.2.4 The continuous function calculus. Let A be a unital C^*-algebra. To each normal element a in A there is one and only one *-isomorphism

$$C(\text{sp}(a)) \to C^*(a, 1) \subseteq A, \qquad f \mapsto f(a),$$

which maps ι to a, where ι in $C(\text{sp}(a))$ is given by $\iota(z) = z$ for all z in $\text{sp}(a)$. This unique *-isomorphism has the properties that its assignment of $f(a)$ agrees with the usual definition when f is a polynomial, and that $\tau(a) = a^*$ when $\tau \colon \mathbb{C} \to \mathbb{C}$ is the function $\tau(z) = \bar{z}$.

The *spectral mapping theorem* says that $\text{sp}(f(a)) = f(\text{sp}(a))$ for every normal element a and every continuous function f on $\text{sp}(a)$.

If $\varphi \colon A \to B$ is a unital *-homomorphism between the C^*-algebras A and B, and if a is a normal element in A, then $\text{sp}(\varphi(a)) \subseteq \text{sp}(a)$, and $f(\varphi(a)) = \varphi(f(a))$ for every f in $C(\text{sp}(a))$.

If a is a normal element in a non-unital C^*-algebra A, then $f(a)$ is, a priori, defined to be an element in the unitization $\widetilde{A}$ of A. With $\pi \colon \widetilde{A} \to \mathbb{C}$ the quotient mapping, since $\pi(f(a)) = f(\pi(a)) = f(0)$ when a belongs to A, we see that $f(a)$ belongs to A precisely when $f(0) = 0$.

In Chapter 2 we shall need the following result.

Lemma 1.2.5. *Let K be a non-empty compact subset of $\mathbb{R}$, and let $f \colon K \to \mathbb{C}$ be a continuous function. Let A be a unital C^*-algebra, and let Ω_K be the set of self-adjoint elements in A with spectrum contained in K. The (induced) function*

$$f \colon \Omega_K \to A, \qquad a \mapsto f(a),$$

is continuous.

Proof. The map $A \to A$ given by $a \mapsto a^n$ is continuous for every integer $n \geqslant 0$ (because multiplication is continuous). It follows that every complex polynomial f induces a continuous map $A \to A$ given by $a \mapsto f(a)$.

Now, let $f \colon K \to \mathbb{C}$ be any continuous function, let a be an element in Ω_K, and let $\varepsilon > 0$. By the Stone–Weierstrass theorem there is a complex polynomial g such that $|f(z) - g(z)| \leqslant \varepsilon/3$ for every z in K. Find $\delta > 0$ such that $\|g(a) - g(b)\| \leqslant \varepsilon/3$ whenever b is an element in A with $\|a - b\| \leqslant \delta$. Since

$$\|f(c) - g(c)\| = \|(f - g)(c)\| = \sup\{|(f - g)(z)| : z \in \text{sp}(c)\} \leqslant \varepsilon/3$$

for all c in Ω_K, it follows that $\|f(a) - f(b)\| \leqslant \varepsilon$ for all b in Ω_K with $\|a - b\| \leqslant \delta$. □

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A more elaborate argument shows that the conclusion of Lemma 1.2.5 holds in the more general situation where K is any non-empty (compact or not) subset of $\mathbb{C}$, and where Ω_K is the set of all normal elements in A with spectrum contained in K.

1.3 Matrix algebras

For each C^*-algebra A and for each natural number n, let $M_n(A)$ be the set of all $n \times n$ matrices

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where each matrix entry a_{ij} belongs to A. Equip $M_n(A)$ with the usual entry-wise vector space operations and matrix multiplication, and set

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & a_{21}^* & \cdots & a_{n1}^* \\ a_{12}^* & a_{22}^* & \cdots & a_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}^* & a_{2n}^* & \cdots & a_{nn}^* \end{pmatrix}.$$

To define a C^*-norm on $M_n(A)$, choose a Hilbert space H and an injective *-homomorphism $\varphi\colon A \to B(H)$. Let $\varphi_n\colon M_n(A) \to B(H^n)$ be given by

$$\varphi_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \varphi(a_{11})\xi_1 + \cdots + \varphi(a_{1n})\xi_n \\ \vdots \\ \varphi(a_{n1})\xi_1 + \cdots + \varphi(a_{nn})\xi_n \end{pmatrix}, \quad \xi_j \in H.$$

Define a norm on $M_n(A)$ by $\|a\| = \|\varphi_n(a)\|$ for a in $M_n(A)$. With these operations, $M_n(A)$ becomes a C^*-algebra, the norm is independent of the choice of representation φ provided that it is injective, and

$$\max_{i,j}\{\|a_{ij}\|\} \leqslant \left\| \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right\| \leqslant \sum_{i,j} \|a_{ij}\|. \tag{1.4}$$

A proof of the the inequality above is outlined in Exercise 1.13. It shows for example that a function $f\colon X \to M_n(A)$ is continuous if and only if each entry function $f_{ij}\colon X \to A$ is continuous. We shall occasionally use the abbreviation

$$\operatorname{diag}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

for a *diagonal matrix*, where $a_1, a_2, \dots, a_n$ are in A.

Forming matrix algebras has the functorial property that if A and B are C^*-algebras and if $\varphi\colon A \to B$ is a *-homomorphism, then the map $\varphi_n\colon M_n(A) \to M_n(B)$ given by

$$\varphi_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & \cdots & \varphi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{n1}) & \cdots & \varphi(a_{nn}) \end{pmatrix} \tag{1.5}$$

is a *-homomorphism for each natural number n. We shall often omit the index n and write φ instead of φ_n.

1.4 Exercises

Exercise 1.1. You are given a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ \varphi\ } E \xrightarrow{\ \psi\ } B \longrightarrow 0.$$

In the notation of Paragraph 1.1.5, show that there is a *-isomorphism $\theta\colon E \to A \oplus B$ that makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ \varphi\ } & E & \xrightarrow{\ \psi\ } & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow[\ \iota_A\ ]{} & A \oplus B & \xrightarrow[\ \pi_B\ ]{} & B & \longrightarrow & 0 \end{array}$$

commutative if and only if there is a *-homomorphism $\nu\colon E \to A$ such that $\nu \circ \varphi = \mathrm{id}_A$.

Exercise 1.2. Show that

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\ \iota\ } C([0,1]) \xrightarrow{\ \psi\ } \mathbb{C} \oplus \mathbb{C} \longrightarrow 0$$

is a short exact sequence, where $\psi(f) = (f(0), f(1))$, and show that the sequence does not split.