1

Preliminary results

I assume familiarity with material from a standard course on elementary algebra. A typical text for such a course is Herstein [He]. A few deeper algebraic results are also needed; they can be found for example in Lang [La]. Section 1 lists the elementary group theoretic results assumed and also contains a list of basic notation. Later sections in chapter 1 introduce some terminology and notation from a few other areas of algebra. Deeper algebraic results are introduced when they are needed.

The last section of chapter 1 contains a brief discussion of group representations. The term representation is used here in a more general sense than usual. Namely a representation of a group G will be understood to be a group homomorphism of G into the group of automorphisms of an object X. Standard use of the term representation requires X to be a vector space.

1 Elementary group theory

Recall that a *binary operation* on a set G is a function from the set product $G \times G$ into G. Multiplicative notation will usually be used. Thus the image of a pair (x, y) under the binary operation will be written xy. The operation is *associative* if $(xy)z = x(yz)$ for all x, y, z in G. The operation is *commutative* if $xy = yx$ for all x, y in G. An *identity* for the operation is an element 1 in G such that $x1 = 1x = x$ for all x in G. An operation possesses at most one identity. Given an operation on G possessing an identity 1, an *inverse* for an element x of G is an element y in G such that $xy = yx = 1$. If our operation is associative and x possesses an inverse then that inverse is unique and is denoted by x^{-1} in multiplicative notation.

A *group* is a set G together with an associative binary operation which possesses an identity and such that each element of G possesses an inverse. The group is *abelian* if its operation is commutative. In the remainder of this section G is a group written multiplicatively.

Let $x \in G$ and n a positive integer. x^n denotes the product of x with itself n times. Associativity insures x^n is a well-defined element of G. Define x^{-n} to be $(x^{-1})^n$ and x^0 to be 1. The usual rules of exponents can be derived from this definition:

(1.1) Let G be a group, $x \in G$, and n and m integers. Then
(1) $(x^n)(x^m) = x^{n+m} = (x^m)(x^n)$.
(2) $(x^n)^m = x^{nm}$.

A *subgroup* of G is a nonempty subset H of G such that for each $x, y \in H$, xy and x^{-1} are in H. This insures that the binary operation on G restricts to a binary operation on H which makes H into a group with the same identity as G and the same inverses. I write $H \leq G$ to indicate that H is a subgroup of G.

(1.2) The intersection of any set of subgroups of G is also a subgroup of G.

Let $S \subseteq G$ and define

$$\langle S \rangle = \bigcap_{S \subseteq H \leq G} H$$

By 1.2, $\langle S \rangle$ is a subgroup of G and by construction it is the smallest subgroup of G containing S. The subgroup $\langle S \rangle$ is called the *subgroup of G generated by S*.

(1.3) Let $S \subseteq G$. Then

$$\langle S \rangle = \{(s_1)^{\varepsilon_1} \ldots (s_n)^{\varepsilon_n} : s_i \in S, \quad \varepsilon = +1 \text{ or } -1\}.$$

(1.4) Let $x \in G$. Then $\langle x \rangle = \{x^n : n \in \mathbb{Z}\}$.

Of course 1.4 is a special case of 1.3. A group G is *cyclic* if it is generated by some element x. In that case x is said to be a *generator* of G and by 1.4, G consists of the powers of x.

The *order* of a group G is the cardinality of the associated set G. Write $|G|$ for the order of a set G or a group G. For $x \in G$, $|x|$ denotes $|\langle x \rangle|$ and is called the *order* of x.

A *group homomorphism* from a group G into a group H is a function $\alpha: G \to H$ of the set G into the set H which preserves the group operations: that is for all x, y in G, $(xy)\alpha = x\alpha y\alpha$. Notice that I usually write my maps on the right, particularly those that are homomorphisms. The homomorphism α is an *isomorphism* if α is a bijection. In that case α possesses an inverse function $\alpha^{-1}: H \to G$ and it turns out α^{-1} is also a group homomorphism. G is *isomorphic* to H if there exists an isomorphism of G and H. Write $G \cong H$ to indicate that G is isomorphic to H. Isomorphism is an equivalence relation. H is said to be a *homomorphic image* of G if there is a surjective homomorphism of G onto H.

A subgroup H of G is *normal* if $g^{-1}hg \in H$ for each $g \in G$ and $h \in H$. Write $H \trianglelefteq G$ to indicate H is a normal subgroup of G. If $\alpha: G \to X$ is a group homomorphism then the *kernel* of α is $\ker(\alpha) = \{g \in G: g\alpha = 1\}$ and it turns out that $\ker(\alpha)$ is a normal subgroup of G. Also write $G\alpha$ for the image $\{g\alpha: g \in G\}$ of G in X. $G\alpha$ is a subgroup of X.

Let $H \leq G$. For $x \in G$ write $Hx = \{hx: h \in H\}$ and $xH = \{xh: h \in H\}$. Hx and xH are *cosets* of H in G. Hx is a right coset and xH a left coset.

To be consistent I'll work with right cosets Hx in this section. G/H denotes the set of all (right) cosets of H in G. G/H is the *coset space* of H in G. Denote by $|G:H|$ the order of the coset space G/H. As the map $h \mapsto hx$ is a bijection of H with Hx, all cosets have the same order, so

(1.5) (Lagrange's Theorem) Let G be a group and $H \leq G$. Then $|G| = |H|\,|G:H|$. In particular if G is finite then $|H|$ divides $|G|$.

If $H \trianglelefteq G$ the coset space G/H is made into a group by defining multiplication via

$$(Hx)(Hy) = Hxy \quad x, y \in G$$

Moreover there is a natural surjective homomorphism $\pi: G \to G/H$ defined by $\pi: x \mapsto Hx$. Notice $\ker(\pi) = H$. Conversely if $\alpha: G \to L$ is a surjective homomorphism with $\ker(\alpha) = H$ then the map $\beta: Hx \mapsto x\alpha$ is an isomorphism of G/H with L such that $\pi\beta = \alpha$. The group G/H is called the *factor group* of G by H. Therefore the factor groups of G over its various normal subgroups are, up to isomorphism, precisely the homomorphic images of G.

(1.6) Let $H \trianglelefteq G$. Then the map $L \mapsto L/H$ is a bijection between the set of all subgroups of G containing H and the set of all subgroups of G/H. Normal subgroups correspond to normal subgroups under this bijection.

For $x, y \in G$, set $x^y = y^{-1}xy$. For $X \subseteq G$ set $X^y = \{x^y: x \in X\}$. X^y is the *conjugate* of X under y. Write X^G for the set $\{X^g: g \in G\}$ of conjugates of X under G. Define

$$N_G(X) = \{g \in G: X^g = X\}.$$

$N_G(X)$ is the *normalizer in G of X* and is a subgroup of G. Indeed if $X \leq G$ then $N_G(X)$ is the largest subgroup of G in which X is normal. Define

$$C_G(X) = \{g \in G: xg = gx \quad \text{for all} \quad x \in X\}.$$

$C_G(X)$ is the *centralizer in G of X*. $C_G(X)$ is also a subgroup of G.

For $X, Y \subseteq G$ define $XY = \{xy : x \in X, y \in Y\}$. The set XY is the *product* of X with Y.

(1.7) Let $X, Y \leq G$. Then

(1) XY is a subgroup of G if and only if $XY = YX$.
(2) If $Y \leq N_G(X)$ then XY is a subgroup of G and $XY/X \cong Y/(Y \cap X)$.
(3) $|XY| = |X||Y|/|X \cap Y|$.

(1.8) Let H and K be normal subgroups of G with $K \leq H$. Then $G/K/H/K \cong G/H$.

Let $G_1, \ldots, G_n$ be a finite set of groups. The *direct product* $G_1 \times \cdots \times G_n = \prod_{i=1}^n G_i$ of the groups $G_1, \ldots, G_n$ is the group defined on the set product $G_1 \times \cdots \times G_n$ by the operation

$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n) \quad x_i, y_i \in G_i$$

(1.9) Let G be a group and $(G_i : 1 \leq i \leq n)$ a family of subgroups of G. Then the following are equivalent:

(1) The map $(x_1, \ldots, x_n) \mapsto x_1 \ldots x_n$ is an isomorphism of G with $G_1 \times \cdots \times G_n$.
(2) $G = \langle G_i : 1 \leq i \leq n \rangle$ and for each i, $1 \leq i \leq n$, $G_i \trianglelefteq G$ and $G_i \cap \langle G_j : j \neq i \rangle = 1$.
(3) $G_i \trianglelefteq G$ for each i, $1 \leq i \leq n$, and each $g \in G$ can be written uniquely as $g = x_1 \ldots x_n$ with $x_i \in G_i$.

If any of the equivalent conditions of 1.9 hold, G will be said to be the *direct product of the subgroups* $(G_i : 1 \leq i \leq n)$.

(1.10) Let $G = \langle g \rangle$ be a cyclic group and $\mathbb{Z}$ the group of integers under addition. Then

(1) If H is a nontrivial subgroup of $\mathbb{Z}$ then $H = \langle n \rangle$, where n is the least positive integer in H.

(2) The map $\alpha : \mathbb{Z} \to G$ defined by $m\alpha = g^m$ is a surjective homomorphism with kernel $\langle n \rangle$, where $n = 0$ if g is of infinite order and $n = \min\{m > 0 : g^m = 1\}$ if g has finite order.

(3) If g has finite order n then $G = \{g^i : 0 \leq i < n\}$ and n is the least positive integer m with $g^m = 1$.

(4) Up to isomorphism $\mathbb{Z}$ is the unique infinite cyclic group and for each positive integer n, the group $\mathbb{Z}_n$ of integers modulo n is the unique cyclic group of order n.

(5) Let $|g| = n$. Then for each divisor m of n, $\langle g^{n/m} \rangle$ is the unique subgroup of G of order m. In particular subgroups of cyclic groups are cyclic.

(1.11) Each finitely generated abelian group is the direct product of cyclic groups.

Let p be a prime. A *p-group* is a group whose order is a power of p. More generally if π is a set of primes then a *π-group* is a group G of finite order such that $\pi(G) \subseteq \pi$, where $\pi(G)$ denotes the set of prime divisors of $|G|$. p' denotes the set of all primes distinct from p. An element x in a group G is a *π-element* if $\langle x \rangle$ is a π-group. An *involution* is an element of order 2.

(1.12) Let $1 \neq G$ be an abelian p-group. Then G is the direct product of cyclic subgroups $G_i \cong \mathbb{Z}_{p^{e_i}}$, $1 \leq i \leq n$, $e_1 \geq e_2 \geq \cdots \geq e_n > 1$. Moreover the integers n and $(e_i : 1 \leq i \leq n)$ are uniquely determined by G.

The *exponent* of a finite group G is the least common multiple of the orders of the elements of G. An *elementary abelian p-group* is an abelian p-group of exponent p. Notice that by 1.12, G is an elementary abelian p-group of order p^n if and only if G is the direct product of n copies of $\mathbb{Z}_p$. In particular up to isomorphism there is a unique elementary abelian p-group of order p^n, which will be denoted by E_{p^n}. The integer n is the *p-rank* of E_{p^n}. The *p-rank* of a general finite group G is the maximum p-rank of an elementary abelian p-subgroup of G, and is denoted by $m_p(G)$.

(1.13) Each group of exponent 2 is abelian.

If π is a set of primes and G a finite group, write $O_\pi(G)$ for the largest normal π-subgroup of G, and $O^\pi(G)$ for the smallest normal subgroup H of G such that G/H is a π-group. $O_\pi(G)$ and $O^\pi(G)$ are well defined by Exercise 1.1.

Define $Z(G) = C_G(G)$ and call $Z(G)$ the *center* of G. If G is a p-group then define

$$\Omega_n(G) = \langle x \in G : x^{p^n} = 1 \rangle$$

$$\mho^n(G) = \langle x^{p^n} : x \in G \rangle.$$

For $X \leq G$ define $\mathrm{Aut}_G(X) = N_G(X)/C_G(X)$ to be the *automizer* in G of X. Notice that by Exercise 1.3, $\mathrm{Aut}_G(X) \leq \mathrm{Aut}(X)$ and indeed $\mathrm{Aut}_G(X)$ is the group of automorphisms induced on X in G.

A *maximal subgroup* of a group G is a proper subgroup of G which is properly contained in no proper subgroup of G. That is a maximal subgroup is a maximal member of the set of proper subgroups of G, partially ordered by inclusion.

If $\alpha: S \to T$ is a function and $R \subseteq S$ then $\alpha|_R$ denotes the restriction of α to R. That is $\alpha|_R: R \to T$ is the function from R into T agreeing with α.

Here's a little result that's easy to prove but useful.

(1.14) (Modular Property of Groups) Let A, B, and C be subgroups of a group G with $A \leq C$. Then $AB \cap C = A(B \cap C)$.

If G is a group write $G^\#$ for the set $G - \{1\}$ of nonidentity elements of G. On the other hand if R is a ring define $R^\# = R - \{0\}$.

Denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$ the complex numbers, the reals, and the rationals, respectively. Often $\mathbb{Z}$ will denote the integers.

Given a group G, a subgroup H of G, and a collection C of subgroups of G, I'll often write $C \cap H$ for the set of members of C which are subgroups of H.

I'll use the *bar convention*. That is I'll often denote a homomorphic image $G\alpha$ of a group G by $\bar{G}$ (or G^* or $\tilde{G}$) and write $\bar{g}$ (or g^* or $\tilde{g}$) for $g\alpha$. This will be done without comment.

Other notation and terminology are introduced in later chapters. The List of Symbols gives the page number where a notation is first introduced and defined.

2 Categories

It will be convenient to have available some of the elementary concepts and language of categories. For a somewhat more detailed discussion, see chapter 1 of Lang [La].

A *category* $\mathscr{C}$ consists of

(1) A collection $\mathrm{Ob}(\mathscr{C})$ of *objects*.
(2) For each pair A,B of objects, a set $\mathrm{Mor}(A,B)$ of *morphisms* from A to B.
(3) For each triple A, B, C of objects a map

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$$

called *composition*. Write fg for the image of the pair (f, g) under the composition map.

Moreover the following three axioms are required to hold:

Cat (1) For each quadruple A, B, C, D of objects, $\mathrm{Mor}(A, B) \cap \mathrm{Mor}(C, D)$ is empty unless $A = C$ and $B = D$.
Cat (2) Composition is associative.
Cat (3) For each object A, $\mathrm{Mor}(A, A)$ possesses an *identity morphism* 1_A such that for all objects B and all f in $\mathrm{Mor}(A, B)$ and g in $\mathrm{Mor}(B, A)$, $1_A f = f$ and $g 1_A = g$.

Almost all categories considered here will be categories of *sets with structure*. That is the objects of the category are sets together with some extra structure, $\mathrm{Mor}(A, B)$ consists of all functions from the set associated to A to the set associated to B which preserve the extra structure, and composition is ordinary composition of functions. The identity morphism 1_A is forced to be the identity map on A. Thus we need to know the identity map preserves structure. We also need to know the composition of maps which preserve structure also preserves structure. These facts will usually be obvious in the examples we consider.

We'll be most interested in the following three categories, which are all categories of sets with structure.

(1) The category of sets and functions: Here the objects are the sets and $\mathrm{Mor}(A, B)$ is the set of all functions from the set A into the set B.
(2) The category of groups and group homomorphisms: The objects are the groups and morphisms are the group homomorphisms.
(3) The category of vector spaces and linear transformations: Fix a field F. The objects are the vector spaces over F and the morphisms are the F-linear transformations.

Let f be a morphism from an object A to an object B. An *inverse* for f in $\mathscr{C}$ is a morphism $g \in \mathrm{Mor}(B, A)$ such that $1_A = fg$ and $1_B = gf$. The morphism f is an *isomorphism* if it possesses an inverse in $\mathscr{C}$. An *automorphism* of A is an isomorphism from A to A. Denote by $\mathrm{Aut}(A)$ the set of all automorphisms of A and observe $\mathrm{Aut}(A)$ forms a group under the composition in $\mathscr{C}$.

If $\alpha: A \to B$ is an isomorphism define $\alpha^*: \mathrm{Mor}(A, A) \to \mathrm{Mor}(B, B)$ by $\beta \to \alpha^{-1}\beta\alpha$ and observe α^* restricts to a group isomorphism of $\mathrm{Aut}(A)$ with $\mathrm{Aut}(B)$.

Let $(A_i : i \in I)$ be a family of objects in a category $\mathscr{C}$. A *coproduct* of the family is an object C together with morphisms $c_i: A_i \to C$, $i \in I$, satisfying the universal property: whenever X is an object and $\alpha_i: A_i \to X$ are morphisms, there exists a unique morphism $\alpha: C \to X$ with $c_i\alpha = \alpha_i$ for each $i \in I$. As a consequence of the universal property, the coproduct of a family is determined up to isomorphism, if it exists.

The *product* of the family is defined dually. That is to obtain the definition of the product, take the definition of the coproduct and reverse the direction of all arrows.

Exercise 1.2 gives a description of coproducts and products in the three categories listed above.

3 Graphs and geometries

This section contains a brief discussion of two more categories which will make occasional appearances in these notes.

A *graph* $\mathscr{G} = (V, *)$ consists of a set V of *vertices* (or objects or points) together with a symmetric relation $*$ called *adjacency* (or incidence or something else). The ordered pairs in the relation are called the *edges* of the graph. I write $u * v$ to indicate two vertices are related via $*$ and say u is *adjacent* to v. A *path of length n* from u to v is a sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ such that $u_i * u_{i+1}$ for each i. Denote by $d(u, v)$ the minimal length of a path from u to v. If no such path exists set $d(u, v) = \infty$. $d(u, v)$ is the *distance* from u to v.

The relation $\sim$ on V defined by $u \sim v$ if and only if $d(u, v) < \infty$ is an equivalence relation on V. The equivalence classes of this relation are called the *connected components* of the graph. The graph is *connected* if it has just one connected component. Equivalently there is a path between any pair of vertices.

A morphism $\alpha: \mathscr{G} \to \mathscr{G}'$ of graphs is a function $\alpha: V \to V'$ from the vertex set V of $\mathscr{G}$ to the vertex set V' of $\mathscr{G}'$ which preserves adjacency; that is if u and v are vertices adjacent in $\mathscr{G}$ then $u\alpha$ is adjacent to $v\alpha$ in $\mathscr{G}'$.

So much for graphs; on to geometries. In this book I adopt a notion of geometry due to Tits. Let I be a finite set. A *geometry* over I is a triple $(\Gamma, \tau, *)$ where Γ is a set of objects, $\tau: \Gamma \to I$ is a type function, and $*$ is a symmetric incidence relation on Γ such that objects u and v of the same type are incident if and only if $u = v$. $\tau(u)$ is the *type* of the object u. Notice $(\Gamma, *)$ is a graph. I'll usually write Γ for the geometry $(\Gamma, \tau, *)$.

A *morphism* $\alpha: \Gamma \to \Gamma'$ of geometries is a function $\alpha: \Gamma \to \Gamma'$ of the associated object sets which preserves type and incidence; that is if $u, v \in \Gamma$ with $u * v$ then $\tau(u) = \tau'(u\alpha)$ and $u\alpha *' v\alpha$.

A *flag* of the geometry Γ is a set T of objects such that each pair of objects in T is incident. Notice our one (weak) axiom insures that a flag T possesses at most one object of each type, so that the type function τ induces an injection of T into I. The image $\tau(T)$ is called the *type* of T. The *rank* and *corank* of T are the order of $\tau(T)$ and $I - \tau(T)$, respectively. The *residue* Γ_T of the flag T is $\{v \in \Gamma - T: v * t \text{ for all } t \in T\}$ regarded as a geometry over $I - \tau(T)$.

The geometry Γ is *connected* if its graph $(\Gamma, *)$ is connected. Γ is *residually connected* if the residue of every flag of corank at least 2 is connected and the residue of every flag of corank 1 is nonempty.

Here's a way to associate geometries to groups. Let G be a group and $\mathscr{F} = (G_i: i \in I)$ a family of subgroups of G. Define $\Gamma(G, \mathscr{F})$ to be the geometry whose set of objects of type i is the coset space G/G_i and with objects $G_i x$ and $G_j y$ incident if $G_i x \cap G_j y$ is nonempty. For $J \subseteq I$ write J' for the complement $I - J$ of J in I and define $G_J = \bigcap_{j \in J} G_j$. Observe that for $x \in G$, $S_{J,x} = \{G_j x: j \in J\}$ is a flag of $\Gamma(G, \mathscr{F})$ of type J.

A group H of automorphisms of a geometry Γ is said to be *flag transitive* if H is transitive on flags of type J for each subset J of I.

4 Abstract representations

Let $\mathscr{C}$ be a category. A *$\mathscr{C}$-representation* of a group G is a group homomorphism $\pi: G \to \text{Aut}(X)$ of G into the group $\text{Aut}(X)$ of automorphisms of some object X in $\mathscr{C}$. (Recall the definition of $\text{Aut}(X)$ in section 2.) We will be most concerned with the following three classes of representations.

A *permutation representation* is a representation in the category of sets and functions. The group $\text{Aut}(X)$ of automorphisms of set X is the *symmetric group* $\text{Sym}(X)$ of X. That is $\text{Sym}(X)$ is the group of all permutations of X under composition.

A *linear representation* is a representation in the category of vector spaces and linear transformations. $\text{Aut}(X)$ is the *general linear group* $GL(X)$ of the vector space X. That is $GL(X)$ is the group of all invertible linear transformations of X.

Finally we will of course be interested in the category of groups and group homomorphisms. Of particular interest is the representation of G via conjugation on itself (cf. Exercise 1.3).

Two $\mathscr{C}$-representations $\pi_i: G \to \text{Aut}(X_i)$, $i = 1, 2$, are said to be *equivalent* if there exists an isomorphism $\alpha: X_1 \to X_2$ such that $\pi_2 = \pi_1\alpha^*$, where $\alpha^*: \text{Aut}(X_1) \to \text{Aut}(X_2)$ is the isomorphism described in section 2. The map α is said to be an *equivalence* of the representations. $\mathscr{C}$-representations $\pi_i: G_i \to \text{Aut}(X_i)$ $i = 1, 2$, are said to be *quasiequivalent* if there exists a group isomorphism $\beta: G_2 \to G_1$ of groups and a $\mathscr{C}$-isomorphism $\alpha: X_1 \to X_2$ such that $\pi_2 = \beta\pi_1\alpha^*$.

Equivalent representations of a group G are the same for our purposes. Quasiequivalent representations are almost the same, differing only by an automorphism of G.

A representation π of G is *faithful* if π is an injection. In that event π induces an isomorphism of G with the subgroup $G\pi$ of $\text{Aut}(X)$, so G may be regarded as a group of automorphisms of X via π.

Let $\pi_i: G \to \text{Aut}(X_i)$, $i = 1, 2$, be $\mathscr{C}$-representations. Define a *G-morphism* $\alpha: X_1 \to X_2$ to be a morphism α of X_1 to X_2 which commutes with the action of G in the sense that $(g\pi_1)\alpha = \alpha(g\pi_2)$ for each $g \in G$. Write $\text{Mor}_G(X_1, X_2)$ for the set of G-morphisms of X_1 to X_2. Notice that the composition of G-morphisms is a G-morphism and the identity morphism is a G-morphism. Similarly define a *G-isomorphism* to be a G-morphism which is also an isomorphism. Notice the G-isomorphisms are the equivalences of representations of G.

One focus of this book is the decomposition of a representation π into smaller representations. Under suitable finiteness conditions (which are always present in the representations considered here) this process of decomposition must terminate, at which point we have associated to π certain indecomposable or irreducible representations which cannot be broken down further. It will develop that the indecomposables associated to π are determined up to equivalence. Thus we are reduced to a consideration of indecomposable representations.

In general indecomposables are not irreducible, so an indecomposable representation π can be broken down further, and we can associate to π a set of irreducible constituents. Sometimes these irreducible constituents are determined up to equivalence, and sometimes not. Even when the irreducible constituents are determined, they usually do not determine π. Thus we will also be concerned with the *extension problem*: Given a set S of irreducible representations, which representations have S as their set of irreducible constituents? There is also the problem of determining the irreducible and indecomposable representations of the group.

It is possible to give a categorical definition of indecomposability (cf. Exercise 1.5). There is also a uniform definition of irreducibility for the classes of representations considered most frequently (cf. Exercise 1.6). I have chosen however to relegate these definitions to the exercises and to make the appropriate definitions of indecomposability and irreducibility for each category in the chapter discussing the elementary representation theory of the category. This process begins in the next chapter, which discusses permutation representations.

However one case is of particular interest. A representation of a group G on itself via conjugation (in the category of groups and group homomorphisms) is irreducible if G possesses no nonidentity proper normal subgroups. In this case G is said to be *simple*. To my mind the simple groups and their irreducible linear and permutation representations are the center of interest in finite group theory.

Exercises for chapter 1

1. Let G be a finite group, π a set of primes, Ω the set of normal π-subgroups of G, and Γ the set of normal subgroups X of G with G/X a π-group. Prove
   (1) If $H, K \in \Omega$ then $HK \in \Omega$. Hence $\langle \Omega \rangle$ is the unique maximal member of Ω.
   (2) If $H, K \in \Gamma$ then $H \cap K \in \Gamma$. Hence $\bigcap_{H \in \Gamma} H$ is the unique minimal member of Γ.
2. Let $\mathscr{C}_1$ be the category of sets and functions, $\mathscr{C}_2$ the category of vector spaces and linear transformations, and $\mathscr{C}_3$ the category of groups and homomorphisms. Let $F = (A_i : 1 \leq i \leq n)$ be a family of objects in $\mathscr{C}_k$. Prove
   (1) Let $k = 1$. Then the coproduct C of F is the disjoint union of the sets A_i with $c_i : A_i \to C$ the inclusion map. The product P of F is