Introduction to Lattices and Order

Introduction to Lattices and Order
Second edition

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Preface to the second edition

This new edition of *Introduction to Lattices and Order* is substantially different from the original one published in 1990. We believe that the revision greatly enhances the book's usefulness and topicality. Our overall aims however remain the same: to provide a textbook introduction which shows the importance of the concept of order in algebra, logic, computer science and other fields and which makes the basic theory accessible to undergraduate and beginning graduate students in mathematics and to professionals in adjacent areas.

In preparing the new edition we have drawn extensively on our teaching experience over the past 10 years and on helpful comments from colleagues. We have taken account of important developments in areas of application, in particular in computer science. Almost all the original material is included, but it has been completely re-organized. Some new material has been added, most notably on Galois connections and fixpoint calculus, and there are many new exercises.

Our objectives in re-arranging the material have been:

- to present elementary and motivational topics as early as possible, for pedagogical reasons;
- to arrange the chapters so that the first part of the book contains core material, suitable for a short, first course;
- to make it easy for particular interest groups to pick out just the sections they want.

Originally, we treated ordered sets first and began the algebraic theory of lattices only in Chapter 5. This meant that some quite sophisticated and specialized material appeared early on, in particular the treatment of CPOs, algebraic lattices and domains. We have now reversed this, and have also made the treatment of the latter topics more independent. We have moved forward the presentation of formal concept analysis so that it now provides a concrete, application-oriented introduction to complete lattices, to which the material on Galois connections and on completions is later linked. There are numerous more localized repackagings of individual topics too, giving a smoother presentation overall. Readers of the first edition who look at the new table of contents will appreciate how major the re-organization is.

Mathematical modelling in computer science has advanced extremely rapidly in the last decade, and this is reflected in the book. We draw attention in particular to:

- our acknowledgement of the importance of Galois connections in formal methods for program development and verification, and
- the revised presentation of fixpoint theorems.

Our debt to those who have pioneered these advances will be clear from the extent to which we have updated the appendix (now Appendix B) which gives suggestions for further reading. Many colleagues, in particular past and present members of the Oxford University Computing Laboratory, have assisted us, either by the insights their books and papers have provided or through their comments. They are too numerous for us to acknowledge their influence and their contributions individually here.

We are grateful to many readers of the first edition who drew our attention to typographical and other minor errors. They were rewarded with a Mars Bar for each misprint found and their corrections were incorporated into the 1994 printing. For the present edition, our sincere thanks go to a team of proof-readers based at La Trobe University: Jane Pitkethly, aided by Miroslav Haviar, Shamsun Naher and Rashed Talukder. They have done a very careful job of eradicating errors that crept into successive drafts, pinpointing obscurities and spotting a few typos from the first edition that were previously missed.

B.A.D. and H.A.P.

June 2001

Preface to the first edition

This is the first textbook devoted to ordered sets and lattices and to their contemporary applications. It acknowledges the increasingly major role order theory is playing on the mathematical stage and is aimed at students of mathematics and at professionals in adjacent areas, including logic, discrete mathematics and computer science.

Lattice theory has been taught to undergraduates at La Trobe University since 1975, and more recently at Oxford University. The notes for these courses were our starting point. The core of the book – Chapters 1, 2 and 5 to 8 – provides a basic introduction to ordered sets, lattices and Boolean algebras and is buttressed by exercises which have been classroom-tested over many years. In a proselytizing article, *Order: a theory with a view* [in *Klassifikation und Ordnung*, INDEKS, Frankfurt, 1989], Ivan Rival discusses the modern role of order. The pictorial philosophy he advocates is strongly evident in our approach: diagrams and diagrammatic arguments are stressed in both the text and the exercises.

Prerequisites are minimal. A reader who has taken a course in linear algebra, group theory or discrete mathematics should have sufficient background knowledge and be familiar with our vocabulary and with those symbols not listed in the notation index. To keep the treatment as elementary as possible, we have denied ourselves the formalism of category theory and of universal algebra. However, we have prepared the ground carefully for those who will progress to texts on general lattice theory or universal algebra and we have included, at the ends of Chapters 2 and 5 and in Chapters 3, 9 and 10, some material suitable for honours students or those beginning graduate work. Inevitably, there was not space for all the topics we should have liked to cover; hints of resisted temptations will be apparent in a few of the exercises. Within lattice theory we have placed the emphasis on distributive lattices. We thereby complement more advanced texts, in which modular and general lattices are already well treated. The study of finite distributive lattices (undertaken in Chapter 8) combines algebraic, order-theoretic and graph-theoretic ideas to provide results which are linked to the ordered set constructions presented in Chapter 1, are easily accessible to undergraduates and are complete in themselves. Our colleagues will doubtless not be surprised that we have also included the extension of the representation theory to the infinite case. To coax those wary of topology, this introduction to duality is accompanied by a self-contained primer containing the small number of topological results which we need.

Order has recently appeared, sometimes a little coyly, in many computational models. The thorough treatment of ordered sets in Chapter 1 (with examples foreshadowing applications in computation) and of intersection structures in Chapter 2 provides a firm foundation on which to build the theory of CPOs and domains. Chapter 3 studies these structures and relates them to Scott's information systems. Our account is necessarily brief. Collateral reading of specialized texts, in which the computer science applications are fully developed, may assist those meeting domain theory for the first time. Chapter 4 deals with fixpoint theory (and also discusses the order-theoretic roots of Zorn's lemma). Thus Chapters 1–4 serve as an introduction to order theory for computer scientists, and for mathematicians seeking to enter their world. In Chapter 11 we look outwards in a different direction and present the rudiments of formal concept analysis. This new field has already made an impact on lattice theory and has much to offer to social scientists concerned with data analysis. We acknowledge our debt to the authors of many unpublished notes and manuscripts on computer science and on concept analysis. In particular, course notes by Dana Scott, Samson Abramsky and Bill Roscoe enticed us into previously unfamiliar territory and Jeff Sanders' notes for the hardware course taught to Mathematics and Computation undergraduates in Oxford influenced our treatment of Boolean algebras.

The technological developments of the 1980s have made our collaboration possible. Our respective computers have faithfully worked many nocturnal hours of overtime. Electronic mail has enabled us to communicate almost daily and, in conjunction with TEX, to confer easily on fine points of presentation in a way that would have been impossible with conventional ('snail') mail. TEX has also allowed us to control the final shape of the text and has given the second author in particular innumerable hours of fun and frustration.

Many people deserve our thanks. We are grateful to David Tranah and the staff of Cambridge University Press for their patient assistance and support and to Dorothy Berridge for her help in typing TEX files. Generations of students have provided valuable consumer feedback and Oxford undergraduates Mark Joshi, Graham Pollitt and Andy Sanderson earn a special mention for their proof-reading. Thanks are due to the colleagues we have pestered to read the book in draft: in particular to Michael Albert, Ralph McKenzie and J.B. Nation. We must also thank Rudolf Wille and Bernhard Ganter for their advice on concept analysis. Notwithstanding electronic communication, we greatly benefited from the opportunity to spend a month discussing the book face to face. The second author gratefully acknowledges the financial assistance

of La Trobe University and the hospitality of its mathematics department. Finally, a very big thank you for their support and forbearance goes to the Davey family: wife Helen and children Evan, Owen and Caitlin.

B.A.D. and H.A.P.

September 1989