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A. W. van der Vaart
Frontmatter
More Information

Asymptotic Statistics

This book is an introduction to the field of asymptotic statistics. The treatment is both practical and mathematically rigorous. In addition to most of the standard topics of an asymptotics course, including likelihood inference, *M*-estimation, asymptotic efficiency, *U*-statistics, and rank procedures, the book also presents recent research topics such as semiparametric models, the bootstrap, and empirical processes and their applications.

One of the unifying themes is the approximation by limit experiments. This entails mainly the local approximation of the classical i.i.d. set-up with smooth parameters by location experiments involving a single, normally distributed observation. Thus, even the standard subjects of asymptotic statistics are presented in a novel way.

Suitable as a text for a graduate or Master's level statistics course, this book also gives researchers in statistics, probability, and their applications an overview of the latest research in asymptotic statistics.

A.W. van der Vaart is Professor of Statistics in the Department of Mathematics and Computer Science at the Vrije Universiteit, Amsterdam.

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Asymptotic Statistics

A.W. VAN DER VAART

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To Maryse and Marianne

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Preface

This book grew out of courses that I gave at various places, including a graduate course in the Statistics Department of Texas A&M University, Master's level courses for mathematics students specializing in statistics at the Vrije Universiteit Amsterdam, a course in the DEA program (graduate level) of Université de Paris-sud, and courses in the Dutch AIO-netwerk (graduate level).

The mathematical level is mixed. Some parts I have used for second year courses for mathematics students (but they find it tough), other parts I would only recommend for a graduate program. The text is written both for students who know about the technical details of measure theory and probability, but little about statistics, and vice versa. This requires brief explanations of statistical methodology, for instance of what a rank test or the bootstrap is about, and there are similar excursions to introduce mathematical details. Familiarity with (higher-dimensional) calculus is necessary in all of the manuscript. Metric and normed spaces are briefly introduced in Chapter 18, when these concepts become necessary for Chapters 19, 20, 21 and 22, but I do not expect that this would be enough as a first introduction. For Chapter 25 basic knowledge of Hilbert spaces is extremely helpful, although the bare essentials are summarized at the beginning. Measure theory is implicitly assumed in the whole manuscript but can at most places be avoided by skipping proofs, by ignoring the word "measurable" or with a bit of handwaving. Because we deal mostly with i.i.d. observations, the simplest limit theorems from probability theory suffice. These are derived in Chapter 2, but prior exposure is helpful.

Sections, results or proofs that are preceded by asterisks are either of secondary importance or are out of line with the natural order of the chapters. As the chart in Figure 0.1 shows, many of the chapters are independent from one another, and the book can be used for several different courses.

A unifying theme is approximation by a limit experiment. The full theory is not developed (another writing project is on its way), but the material is limited to the "weak topology" on experiments, which in 90% of the book is exemplified by the case of smooth parameters of the distribution of i.i.d. observations. For this situation the theory can be developed by relatively simple, direct arguments. Limit experiments are used to explain efficiency properties, but also why certain procedures asymptotically take a certain form.

A second major theme is the application of results on abstract empirical processes. These already have benefits for deriving the usual theorems on M-estimators for Euclidean parameters but are indispensable if discussing more involved situations, such as M-estimators with nuisance parameters, chi-square statistics with data-dependent cells, or semiparametric models. The general theory is summarized in about 30 pages, and it is the applications

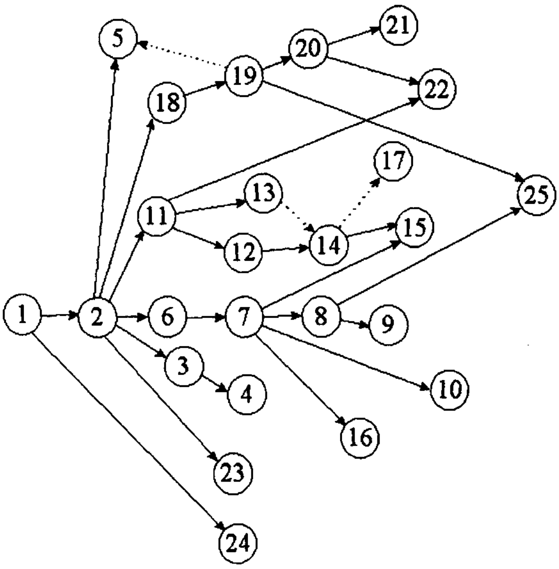

Figure 0.1. Dependence chart. A solid arrow means that a chapter is a prerequisite for a next chapter. A dotted arrow means a natural continuation. Vertical or horizontal position has no independent meaning.

that we focus on. In a sense, it would have been better to place this material (Chapters 18 and 19) earlier in the book, but instead we start with material of more direct statistical relevance and of a less abstract character. A drawback is that a few (starred) proofs point ahead to later chapters.

Almost every chapter ends with a "Notes" section. These are meant to give a rough historical sketch, and to provide entries in the literature for further reading. They certainly do not give sufficient credit to the original contributions by many authors and are not meant to serve as references in this way.

Mathematical statistics obtains its relevance from applications. The subjects of this book have been chosen accordingly. On the other hand, this is a mathematician's book in that we have made some effort to present results in a nice way, without the (unnecessary) lists of "regularity conditions" that are sometimes found in statistics books. Occasionally, this means that the accompanying proof must be more involved. If this means that an idea could go lost, then an informal argument precedes the statement of a result.

This does not mean that I have strived after the greatest possible generality. A simple, clean presentation was the main aim.

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Notation

A^*	adjoint operator
$\mathbb{B}^*$	dual space
$C_b(T)$, $UC(T)$, $C(T)$	(bounded, uniformly) continuous functions on T
$\ell^\infty(T)$	bounded functions on T
$\mathcal{L}_r(Q)$, $L_r(Q)$	measurable functions whose rth powers are Q-integrable
$\lVert f \rVert_{Q,r}$	norm of $L_r(Q)$
$\lVert z \rVert_\infty$, $\lVert z \rVert_T$	uniform norm
lin	linear span
$\mathbb{C}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$	number fields and sets
$\mathrm{E}X$, E^*X, $\mathrm{var}\,X$, $\mathrm{sd}\,X$, $\mathrm{Cov}\,X$	(outer) expectation, variance, standard deviation, covariance (matrix) of X
$\mathbb{P}_n$, $\mathbb{G}_n$	empirical measure and process
$\mathbb{G}_P$	P-Brownian bridge
$N(\mu, \Sigma)$, t_n, χ^2_n	normal, t and chisquare distribution
z_α, $\chi^2_{n,\alpha}$, $t_{n,\alpha}$	upper α-quantiles of normal, chisquare and t distributions
$\ll$	absolutely continuous
$\lhd$, $\lhd \rhd$	contiguous, mutually contiguous
$\lesssim$	smaller than up to a constant
$\rightsquigarrow$	convergence in distribution
$\xrightarrow{\mathrm{P}}$	convergence in probability
$\xrightarrow{\mathrm{as}}$	convergence almost surely
$N(\varepsilon, T, d)$, $N_{[]}(\varepsilon, T, d)$	covering and bracketing number
$J(\varepsilon, T, d)$, $J_{[]}(\varepsilon, T, d)$	entropy integral
$o_P(1)$, $O_P(1)$	stochastic order symbols