Pure Mathematics 4

Hugh Neill and
Douglas Quadling

Series editor Hugh Neill

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2001
Reprinted 2003

Printed in the United Kingdom at the University Press, Cambridge

Typefaces Times, Helvetica *Systems* Microsoft® Word, MathType™

A catalogue record for this book is available from the British Library

ISBN 0 521 78371 2 paperback

Cover image: Images Colour Library

The authors and publisher would like to thank the Nuffield Foundation for providing the artwork which appears in adapted form on page 30.

Contents

1 Rational functions

This chapter takes partial fractions on from P3 Chapter 7. When you have completed it, you should

- be able to put a rational function into partial fractions when one of the factors is a quadratic which does not factorise
- be able to put a rational function into partial fractions when the degree of the numerator is not less than the degree of the denominator.

1.1 Partial fractions

In P3 Chapter 7, you learned how to split rational functions of the forms

$$\frac{ax+b}{(px+q)(rx+s)} \quad \text{and} \quad \frac{ax^2+bx+c}{(px+q)(rx+s)^2}$$

into partial fractions.

Neither of these types includes a form such as $\frac{6x}{(x-1)\left(x^2+2x+3\right)}$, where the quadratic factor x^2+2x+3 in the denominator does not factorise.

In this case, you would certainly expect to write

$$\frac{6x}{(x-1)\left(x^2+2x+3\right)} \equiv \frac{A}{x-1} + \frac{\text{something}}{x^2+2x+3}.$$

But what is the 'something'? Multiply by $x-1$, and put $x=1$ in the subsequent identity.

$$\frac{6x}{\left(x^2+2x+3\right)} \equiv A + \frac{(\text{something})(x-1)}{x^2+2x+3} \text{ with } x=1 \text{ gives } A=1.$$

Therefore $\frac{6x}{(x-1)\left(x^2+2x+3\right)} \equiv \frac{1}{x-1} + \frac{\text{something}}{x^2+2x+3}$.

You can now find what the 'something' is by subtraction, since

$$\frac{\text{something}}{x^2+2x+3} \equiv \frac{6x}{(x-1)\left(x^2+2x+3\right)} - \frac{1}{x-1} \equiv \frac{6x-\left(x^2+2x+3\right)}{(x-1)\left(x^2+2x+3\right)}$$

$$\equiv \frac{-x^2+4x-3}{(x-1)\left(x^2+2x+3\right)} \equiv \frac{-(x-1)(x-3)}{(x-1)\left(x^2+2x+3\right)} \equiv \frac{-x+3}{x^2+2x+3}.$$

The 'something' has the form $Bx+C$.

The method shown in this example will be called Method 1, with the change that $Bx+C$ will be used in place of 'something'.

Here are two other methods which you can use.

If you write $\dfrac{6x}{(x-1)\left(x^2+2x+3\right)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+2x+3}$ initially, you can combine the partial fractions to get a numerator which is a quadratic, so you can equate the coefficients of x^0, x^1 and x^2 to get three equations, and solve them for A, B and C. This is Method 2.

Or you can multiply by $x-1$ and put $x=1$ as before to get $A=1$, and then equate coefficients to find B and C. This is Method 3.

The two examples which follow show these three methods of getting the partial fractions. Use whichever method you find easiest.

Method 1 has the advantage that it is self-checking. If the fraction just before the final result does not cancel, you have made a mistake. If it cancels, there is probably no mistake.

Method 2 is in many ways the most straightforward, but involves solving three equations for A, B and C.

Notice that in Method 3 the coefficients of x^2 and x^0 are used to get the values of B and C. This is because they give the simplest equations. You will usually find that the highest and lowest powers give the simplest equations in these situations. However, you should be aware that equating the coefficients of x^0 has the same result as putting $x=0$; you get no extra information.

Example 1.1.1

Split $\dfrac{5x-4}{(x+2)\left(x^2+3\right)}$ into partial fractions.

Method 1 Write $\dfrac{5x-4}{(x+2)\left(x^2+3\right)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+3}$.

Multiply by $x+2$ to get $\dfrac{5x-4}{\left(x^2+3\right)} \equiv A + \dfrac{(Bx+C)(x+2)}{x^2+3}$, and put $x=-2$. Then $\dfrac{5\times(-2)-4}{(-2)^2+3} = A$, so $A=-2$.

$$\text{Then } \frac{Bx+C}{x^2+3} \equiv \frac{5x-4}{(x+2)\left(x^2+3\right)} - \frac{-2}{x+2}$$

$$\equiv \frac{5x-4-(-2)\left(x^2+3\right)}{(x+2)\left(x^2+3\right)}$$

$$\equiv \frac{2x^2+5x+2}{(x+2)\left(x^2+3\right)} \equiv \frac{(x+2)(2x+1)}{(x+2)\left(x^2+3\right)} \equiv \frac{2x+1}{x^2+3}.$$

So $\dfrac{5x-4}{(x+2)(x^2+3)} \equiv \dfrac{-2}{x+2} + \dfrac{2x+1}{x^2+3}$.

Method 2 Write $\dfrac{5x-4}{(x+2)(x^2+3)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+3}$.

Multiply by $(x+2)(x^2+3)$ to get $5x-4 \equiv A(x^2+3)+(Bx+C)(x+2)$,

which you can write as

$$5x-4 \equiv (A+B)x^2 + (2B+C)x + 3A + 2C.$$

The three equations which you get by equating coefficients of x^0, x^1 and x^2 are

$$3A+2C=-4, \quad 2B+C=5 \quad \text{and} \quad A+B=0.$$

Solving these equations gives $A=-2, \quad B=2 \quad$ and $\quad C=1$, so

$$\frac{5x-4}{(x+2)(x^2+3)} \equiv \frac{-2}{x+2} + \frac{2x+1}{x^2+3}.$$

Method 3 Write $\dfrac{5x-4}{(x+2)(x^2+3)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+3}$.

Multiply by $x+2$ to get $\dfrac{5x-4}{x^2+3} \equiv A + \dfrac{(Bx+C)(x+2)}{x^2+3}$, and put $x=-2$. Then $\dfrac{5\times(-2)-4}{(-2)^2+3} = A$, so $A=-2$.

Combining the fractions in $\dfrac{5x-4}{(x+2)(x^2+3)} \equiv \dfrac{-2}{x+2} + \dfrac{Bx+C}{x^2+3}$ and equating the numerators gives

$$5x-4 \equiv -2(x^2+3)+(Bx+C)(x+2),$$

which you can write as

$$5x-4 \equiv (-2+B)x^2 + (2Bx+C)x - 6 + 2C.$$

Equating the coefficients of x^2 gives $-2+B=0$, so $B=2$.

Equating the coefficients of x^0 gives $-4=-6+2C$, so $C=1$.

Checking the x-coefficient: on the left side 5; on the right $2B+C=2\times2+1=5$.

Therefore $\dfrac{5x-4}{(x+2)(x^2+3)} \equiv \dfrac{-2}{x+2} + \dfrac{2x+1}{x^2+3}$.

Example 1.1.2

Find $\displaystyle\int_2^3 \frac{x+1}{(x-1)\left(x^2+1\right)}\,\mathrm{d}x$.

To integrate you need to put $\dfrac{x+1}{(x-1)\left(x^2+1\right)}$ into partial fractions.

Method 1 Write $\dfrac{x+1}{(x-1)\left(x^2+1\right)} \equiv \dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$.

Multiplying by $x-1$ and putting $x=1$ gives $A=1$.

Calculating $\dfrac{x+1}{(x-1)\left(x^2+1\right)}-\dfrac{1}{x-1}$ to find the other fraction on the right gives

$$\frac{x+1}{(x-1)\left(x^2+1\right)}-\frac{1}{x-1} \equiv \frac{x+1-\left(x^2+1\right)}{(x-1)\left(x^2+1\right)} \equiv \frac{-\left(x^2-x\right)}{(x-1)\left(x^2+1\right)}$$
$$\equiv \frac{-x(x-1)}{(x-1)\left(x^2+1\right)} \equiv -\frac{x}{x^2+1}.$$

Therefore $\dfrac{x+1}{(x-1)\left(x^2+1\right)} \equiv \dfrac{1}{x-1}-\dfrac{x}{x^2+1}$. Then

$$\int_2^3 \frac{x+1}{(x-1)\left(x^2+1\right)}\,\mathrm{d}x = \int_2^3\left(\frac{1}{x-1}-\frac{x}{x^2+1}\right)\mathrm{d}x = \left[\ln(x-1)-\tfrac{1}{2}\ln\left(x^2+1\right)\right]_2^3$$
$$= \left(\ln 2-\tfrac{1}{2}\ln 10\right)-\left(\ln 1-\tfrac{1}{2}\ln 5\right) = \ln 2-\tfrac{1}{2}(\ln 10-\ln 5)$$
$$= \ln 2-\tfrac{1}{2}(\ln 2) = \tfrac{1}{2}\ln 2.$$

Example 1.1.3

Calculate $\displaystyle\int_{-2}^2 \frac{x^2+x}{(x-4)\left(x^2+4\right)}\,\mathrm{d}x$.

Putting $\dfrac{x^2+x}{(x-4)\left(x^2+4\right)}$ into partial fractions gives $\dfrac{x^2+x}{(x-4)\left(x^2+4\right)} \equiv \dfrac{1}{x-4}+\dfrac{1}{x^2+4}$.

So $$\int_{-2}^2 \frac{x^2+x}{(x-4)\left(x^2+4\right)}\,\mathrm{d}x = \int_{-2}^2\left(\frac{1}{x-4}+\frac{1}{x^2+4}\right)\mathrm{d}x$$
$$= \int_{-2}^2 \frac{1}{x-4}\,\mathrm{d}x + \int_{-2}^2 \frac{1}{x^2+4}\,\mathrm{d}x.$$

The first of these integrals is $\left[\ln|x-4|\right]_{-2}^2 = \ln 2-\ln 6 = \ln\tfrac{1}{3} = -\ln 3$.

The second integral is carried out by the substitution $x = 2\tan u$ (P3 Section 10.2).

$$\int_{-2}^{2} \frac{1}{x^2+4}\,dx = \int_{\tan^{-1}(-1)}^{\tan^{-1}1} \frac{1}{4\sec^2 u} \times 2\sec^2 u\,du = \tfrac{1}{2}[u]_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} = \tfrac{1}{4}\pi.$$

Thus $\displaystyle\int_{-2}^{2} \frac{x^2+x}{(x-4)(x^2+4)}\,dx = \tfrac{1}{4}\pi - \ln 3.$

Exercise 1A

1 Express each of the following in partial fractions.

(a) $\dfrac{x^2+x}{(x-1)(x^2+1)}$ (b) $\dfrac{2x+1}{(x+1)(x^2+2x+2)}$ (c) $\dfrac{2x^2+x-3}{(x+3)(x^2+3)}$

(d) $\dfrac{2x^2+15x}{(2x-3)(x^2+x+3)}$ (e) $\dfrac{1+2x^2}{(2+3x)(3+3x+2x^2)}$ (f) $\dfrac{17+10x}{(1+4x)(2+x+x^2)}$

(g) $\dfrac{9x+4}{(x-2)(x^2+3x+1)}$ (h) $\dfrac{9-x-4x^2}{(2x-3)(2x^2-3)}$ (i) $\dfrac{17x+3}{(3x-2)(4x^2+3)}$

(j) $\dfrac{7x-13}{(2x+1)(x^2-x+2)}$ (k) $\dfrac{x^2+25x+12}{(5x-7)(2x^2-2x+5)}$ (l) $\dfrac{11x-2x^2-60}{8x^3+27}$

2 Express each of the following in partial fractions.

(a) $\dfrac{5x^2+3x-3}{(2x-1)(x^2+3x-2)}$ (b) $\dfrac{26x^2-15}{(4x-3)(2x^2+x-2)}$ (c) $\dfrac{4x^2-x+3}{(x-1)(x^2+x+1)}$

(d) $\dfrac{12x^2-15x+10}{(3x-2)(3x^2-x+2)}$ (e) $\dfrac{9-16x}{(2-x)(4x^2+3x+1)}$ (f) $\dfrac{3-4x-x^2}{1+x^3}$

3 Find the values of the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{1-2x-x^2}{(1+x)(1+x^2)}\,dx$ (b) $\displaystyle\int_1^2 \frac{2+3x^2}{x(x^2+2)}\,dx$

(c) $\displaystyle\int_1^3 \frac{4x+9}{(2x-1)(x^2+x+2)}\,dx$ (d) $\displaystyle\int_0^1 \frac{18x^2+10x+7}{(1+3x)(2x^2+x+2)}\,dx$

1.2 Improper fractions

So far all the rational functions you have seen have been ‘proper’ fractions. That is, the degree of the numerator has been less than the degree of the denominator.

Rational functions such as $\dfrac{x^2-3x+5}{(x+1)(x-2)}$ and $\dfrac{x^3}{x^2+1}$, in which the degree of the numerator is greater than or equal to the degree of the denominator, are called **improper fractions**.

Improper fractions are often not the most convenient form to work with; it is frequently better to express them in a different way.

For example, you cannot find $\int \frac{6x}{x-1}\,\mathrm{d}x$ with the integrand in its present form, but it is quite straightforward to integrate the same expression as $\int \left(6+\frac{6}{x-1}\right)\mathrm{d}x$. You get

$$\int \left(6+\frac{6}{x-1}\right)\mathrm{d}x = 6x + 6\ln|x-1| + k.$$

Similarly, if you wish to sketch the graph of $y=\frac{6x}{x-1}$, although you can see immediately that the graph passes through the origin, other features are much clearer in the form $y = 6+\frac{6}{x-1}$. This idea will be taken further in Chapter 4.

You may find it helpful to think about an analogy between improper fractions in arithmetic and improper fractions in algebra. Sometimes in arithmetic it is more useful to think of the number $\frac{25}{6}$ *in that form; at other times it is better in the form* $4\frac{1}{6}$. *The same is true in algebra, and you need to be able to go from one form to the other .*

In P2 Section 1.4, you learned how to divide one polynomial by another polynomial to get a quotient and a remainder.

For example, when you divide the polynomial $\mathrm{a}(x)$ by the polynomial $\mathrm{b}(x)$ you will get a quotient $\mathrm{q}(x)$ and a remainder $\mathrm{r}(x)$ defined by

$$\mathrm{a}(x) \equiv \mathrm{b}(x)\mathrm{q}(x) + \mathrm{r}(x)$$

where the degree of the remainder $\mathrm{r}(x)$ is less than the degree of the divisor $\mathrm{b}(x)$.

If you divide this equation by $\mathrm{b}(x)$, you get

$$\frac{\mathrm{a}(x)}{\mathrm{b}(x)} \equiv \frac{\mathrm{b}(x)\mathrm{q}(x)+\mathrm{r}(x)}{\mathrm{b}(x)} \equiv \mathrm{q}(x) + \frac{\mathrm{r}(x)}{\mathrm{b}(x)}.$$

Therefore, if you divide x^2-3x+5 by $(x+1)(x-2)$ to get a number A and a remainder of the form $Px+Q$, it is equivalent to saying that

$$\frac{x^2-3x+5}{(x+1)(x-2)} \equiv A + \frac{Px+Q}{(x+1)(x-2)}.$$

This form will be called **divided out form**.

The analogous process in arithmetic is to say that 25 *divided by* 6 *is* 4 *with remainder* 1*; that is,* $25 = 4\times 6+1$, *or* $\frac{25}{6} = 4+\frac{1}{6} = 4\frac{1}{6}$.

Example 1.2.1

Express $\dfrac{x^2-3x+5}{(x+1)(x-2)}$ in the form $A+\dfrac{Px+Q}{(x+1)(x-2)}$, by finding A, P and Q.

Writing $\dfrac{x^2-3x+5}{(x+1)(x-2)} \equiv A+\dfrac{Px+Q}{(x+1)(x-2)}$ and multiplying both sides by $(x+1)(x-2)$ gives $x^2-3x+5 \equiv A(x+1)(x-2)+Px+Q$.

Equating the coefficients of x^2 gives $A=1$.

At this stage you can either continue to equate coefficients of other powers or calculate $\dfrac{x^2-3x+5}{(x+1)(x-2)}-1$ to find $\dfrac{Px+Q}{(x+1)(x-2)}$.

Taking the second alternative,

$$\frac{x^2-3x+5}{(x+1)(x-2)}-1 \equiv \frac{x^2-3x+5-(x+1)(x-2)}{(x+1)(x-2)}$$
$$\equiv \frac{x^2-3x+5-\left(x^2-x-2\right)}{(x+1)(x-2)} \equiv \frac{-2x+7}{(x+1)(x-2)}.$$

Therefore $A=1$, $P=-2$ and $Q=7$.

You can see from the form $\dfrac{x^2-3x+5}{(x+1)(x-2)} \equiv 1+\dfrac{-2x+7}{(x+1)(x-2)}$ that, for large values of x, the term on the right gets very small, so $\dfrac{x^2-3x+5}{(x+1)(x-2)}$ gets very close to 1. This point is taken further in Chapter 4.

1.3 Improper fractions and partial fractions

You can now put together the work on improper fractions in Section 1.2 with the work on partial fractions in Section 1.1 and P3 Chapter 7.

For example, if you take the result from Example 1.2.1,

$$\frac{x^2-3x+5}{(x+1)(x-2)} \equiv 1+\frac{-2x+7}{(x+1)(x-2)},$$

you can go further by putting the right side into partial fraction form using the standard method and getting

$$\frac{x^2-3x+5}{(x+1)(x-2)} \equiv 1-\frac{3}{x+1}+\frac{1}{x-2}.$$

Example 1.3.1

Split $\frac{2x^2+4x-3}{(x+1)(2x-3)}$ into partial fractions.

There are a number of methods you can use. You could go immediately into a partial fraction form and find the coefficients either by equating coefficients or by other methods. Or you could put it into divided out form first, and then use one of the standard methods for the remaining partial fractions.

Method A This method goes straight into partial fraction form, but does not use the equating coefficient technique.

Write $\frac{2x^2+4x-3}{(x+1)(2x-3)}$ in the form $\frac{2x^2+4x-3}{(x+1)(2x-3)} \equiv A + \frac{B}{x+1} + \frac{C}{2x-3}$.

Multiplying by $x+1$ gives $\frac{2x^2+4x-3}{2x-3} \equiv A(x+1) + B + \frac{C(x+1)}{2x-3}$.

Putting $x=-1$ gives $\frac{2\times(-1)^2+4\times(-1)-3}{2\times(-1)-3} = B$, so $B=1$.

Multiplying instead by $2x-3$ gives $\frac{2x^2+4x-3}{x+1} \equiv A(2x-3) + \frac{B(2x-3)}{x+1} + C$.

Putting $x=\frac{3}{2}$ gives $\frac{2\times\left(\frac{3}{2}\right)^2+4\times\left(\frac{3}{2}\right)-3}{\left(\frac{3}{2}+1\right)} = C$, so $C=3$.

Therefore $\frac{2x^2+4x-3}{(x+1)(2x-3)} \equiv A + \frac{1}{x+1} + \frac{3}{2x-3}$.

Putting $x=0$ gives $A=1$.

Therefore $\frac{2x^2+4x-3}{(x+1)(2x-3)} \equiv 1 + \frac{1}{x+1} + \frac{3}{2x-3}$.

Method B If you divide out first, you start with the form $\frac{2x^2+4x-3}{(x+1)(2x-3)} \equiv A + \frac{Px+Q}{(x+1)(2x-3)}$. By multiplying by $(x+1)(2x-3)$ and equating the coefficients of x^2, you find that $A=1$.

Thus $\frac{2x^2+4x-3}{(x+1)(2x-3)} \equiv 1 + \frac{Px+Q}{(x+1)(2x-3)}$ so $\frac{2x^2+4x-3}{(x+1)(2x-3)} - 1 \equiv \frac{Px+Q}{(x+1)(2x-3)}$.

Simplifying the left side

$$\frac{2x^2+4x-3-(x+1)(2x-3)}{(x+1)(2x-3)} \equiv \frac{2x^2+4x-3-\left(2x^2-x-3\right)}{(x+1)(2x-3)} \equiv \frac{5x}{(x+1)(2x-3)}.$$

Any of the standard methods for partial fractions now shows that

$$\frac{5x}{(x+1)(2x-3)} \equiv \frac{1}{x+1} + \frac{3}{2x-3}, \text{ so } \frac{2x^2+4x-3}{(x+1)(2x-3)} \equiv 1 + \frac{1}{x+1} + \frac{3}{2x-3}.$$

Use whichever method suits you best. Equating coefficients and solving the resulting equations always works, but it is not always the quickest method.

In Example 1.3.1, the numerator $2x^2+4x-3$ was a polynomial of the same degree as the denominator $(x+1)(2x-3)$, but the numerator could be a polynomial of higher degree than the denominator.

If the polynomial in the numerator of a fraction is of higher degree than the denominator, you need to find the difference in degree. That is, if you divide the numerator by the denominator to get a quotient and a remainder, what will the degree of your quotient be? The answer will tell you what polynomial to write in the divided out form.

Example 1.3.2

Split $\dfrac{3x^4-3x^3-17x^2-x-1}{(x-3)(x+2)}$ into partial fractions.

In this case, the degree of the numerator is 4 and the degree of the denominator is 2. This shows that the degree of the quotient polynomial when you divide the numerator by the denominator is $4-2=2$. Therefore write $\dfrac{3x^4-3x^3-17x^2-x-1}{(x-3)(x+2)}$ in the form

$$\frac{3x^4-3x^3-17x^2-x-1}{(x-3)(x+2)} \equiv Ax^2+Bx+C+\frac{D}{x-3}+\frac{E}{x+2},$$

where the polynomial Ax^2+Bx+C has the degree $4-2=2$.

The process of multiplying by $x-3$, and putting $x=3$ in the resulting identity, as in Example 1.2.1, gives $D=1$. Similarly, multiplying by $x+2$, and then putting $x=-2$ in the resulting identity, as in Example 1.2.1, gives $E=-1$. Therefore

$$\frac{3x^4-3x^3-17x^2-x-1}{(x-3)(x+2)} \equiv Ax^2+Bx+C+\frac{1}{x-3}-\frac{1}{x+2}.$$

It is best now to multiply both sides of the identity by $(x-3)(x+2)$ and to equate coefficients, choosing carefully which powers of x to use to minimise the work.

$$3x^4-3x^3-17x^2-x-1 \equiv \left(Ax^2+Bx+C\right)(x-3)(x+2)+(x+2)-(x-3)$$

Equating the coefficients of x^4: $3=A$.

Equating the coefficients of x^0: $-1=-6C+2+3$, so $C=1$.

Equating the coefficients of x^1: $-1=-6B-C+1-1$, so $B=0$.

Therefore $\dfrac{3x^4-3x^3-17x^2-x-1}{(x-3)(x+2)} \equiv 3x^2+1+\dfrac{1}{x-3}-\dfrac{1}{x+2}$.

Example 1.3.3

Find $\displaystyle\int \frac{x^3-3x-4}{x^3-1}\,dx$.

You cannot integrate this directly, so put it into divided out form to get $\displaystyle\int \frac{x^3-3x-4}{x^3-1}\,dx = \int\left(1+\frac{-3x-3}{x^3-1}\right)dx$. (It is worth doing this first before going into partial fractions, because you might be lucky and be able to integrate the term on the right easily.) In this case you cannot directly integrate the term on the right, so put it into partial fractions, using the fact that $x^3-1\equiv(x-1)\left(x^2+x+1\right)$. Then

$$\frac{-3x-3}{x^3-1}\equiv\frac{-3x-3}{(x-1)\left(x^2+x+1\right)}\equiv\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Multiplying by $x-1$, and putting $x=1$ in the resulting identity, gives $A=-2$.

Therefore $$\frac{Bx+C}{x^2+x+1}\equiv\frac{-3x-3}{(x-1)\left(x^2+x+1\right)}-\frac{-2}{x-1}\equiv\frac{-3x-3+2\left(x^2+x+1\right)}{(x-1)\left(x^2+x+1\right)}$$
$$\equiv\frac{2x^2-x-1}{(x-1)\left(x^2+x+1\right)}\equiv\frac{(x-1)(2x+1)}{(x-1)\left(x^2+x+1\right)}\equiv\frac{2x+1}{x^2+x+1}.$$

Thus $$\int\frac{x^3-3x-4}{x^3-1}\,dx=\int\left(1-\frac{2}{x-1}+\frac{2x+1}{x^2+x+1}\right)dx$$
$$=x-2\ln|x-1|+\ln\left(x^2+x+1\right)+k.$$

You do not need a modulus sign for x^2+x+1 since $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ and is therefore always positive.

Exercise 1B

1 Express the following improper fractions in partial fraction form.

(a) $\dfrac{2x^2+x+1}{(x+1)(x-1)}$ (b) $\dfrac{x^2+1}{(x+3)(x-2)}$ (c) $\dfrac{7-5x+2x^2}{(3-x)(1-2x)}$

(d) $\dfrac{6x^2+11x+3}{(x+2)(2x-1)}$ (e) $\dfrac{4x^3+6x+1}{(x+1)(2x-1)}$ (f) $\dfrac{x\left(17+30x+9x^2\right)}{(3+x)(1+3x)}$

(g) $\dfrac{2x^4+5x^3-2x-2}{(x+2)(2x+1)}$ (h) $\dfrac{4x^4+8x^3-11x^2-36}{(x+3)(2x-3)}$ (i) $\dfrac{3x^4}{x^3+1}$

2 Find the values of the following definite integrals.

(a) $\displaystyle\int_2^3 \frac{x^3+x^2+x-1}{x(x^2-1)}\,dx$ (b) $\displaystyle\int_1^2 \frac{2+2x+2x^2+x^3}{x^2(1+x)}\,dx$

(c) $\displaystyle\int_0^1 \frac{6x^3-7x^2-11}{(2x-3)(x^2+1)}\,dx$ (d) $\displaystyle\int_0^1 \frac{2x^4+3x^3-10x^2-5x+2}{(x+1)(x^2-x-1)}\,dx$

Miscellaneous exercise 1

1 Express the following in partial fractions.

(a) $\dfrac{1+x}{x^2(1-x)}$ (b) $\dfrac{6+x-x^4}{(x+2)(x^2+2)}$

2 (a) Given that

$$\frac{2}{(x-1)^2(x^2+1)} \equiv \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx}{x^2+1},$$

find the values of the constants A, B and C.

(b) Show that $\displaystyle\int_2^3 \frac{2}{(x-1)^2(x^2+1)}\,dx = a+b\ln 2$, where a and b are constants whose values you should find. (OCR)

3 (a) Find the values of A, B and C for which $\dfrac{x^2-2}{(x-2)^2} \equiv A+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$.

(b) The region bounded by the curve with equation $y=\dfrac{x^2-2}{(x-2)^2}$, the x-axis and the lines $x=3$ and $x=4$ is denoted by R. Show that R has area $(2+4\ln 2)$ square units. (OCR)

4 (a) Express $\dfrac{2}{(x-1)(x-3)}$ in partial fractions, and use the result to express $\dfrac{4}{(x-1)^2(x-3)^2}$ in partial fractions.

The finite region bounded by the curve with equation $y=\dfrac{2}{(x-1)(x-3)}$ and the lines $x=4$ and $y=\frac{1}{4}$ is denoted by R.

(b) Show that the area of R is $\ln\frac{3}{2}-\frac{1}{4}$.

(c) Calculate the volume of the solid formed when R is rotated through 2π radians about the x-axis. (OCR)

5 Evaluate $\displaystyle\int_0^1 \frac{x^2+6x-4}{(x+2)(x^2+2)}\,dx$.

6 Calculate the exact value of $\int_0^{\frac{1}{2}} \frac{1+x+x^2}{(1-x)^2}\,dx$.

7 Calculate the exact value of $\int_0^1 \frac{x^2+x+1}{(2x+1)(x+1)^2}\,dx$.

8 (a) Find the value of a such that a translation in the direction of the x-axis transforms the curve with equation $y=\frac{x^2+ax-1}{x^3}$ into the curve with equation $y=\frac{x^2-2}{(x-1)^3}$.

(b) Hence find the exact value of $\int_2^3 \frac{x^2-2}{(x-1)^3}\,dx$. (OCR)