THEORY OF FINANCIAL RISKS
FROM STATISTICAL PHYSICS TO RISK MANAGEMENT

This book summarizes recent theoretical developments inspired by statistical physics in the description of the potential moves in financial markets, and its application to derivative pricing and risk control. The possibility of accessing and processing huge quantities of data on financial markets opens the path to new methodologies where systematic comparison between theories and real data not only becomes possible, but mandatory. This book takes a physicist’s point of view of financial risk by comparing theory with experiment. Starting with important results in probability theory the authors discuss the statistical analysis of real data, the empirical determination of statistical laws, the definition of risk, the theory of optimal portfolio and the problem of derivatives (forward contracts, options). This book will be of interest to physicists interested in finance, quantitative analysts in financial institutions, risk managers and graduate students in mathematical finance.

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Until recently, finance theory appeared to be reaching a triumphant climax. Many years ago, Harry Markowitz and William Sharpe had shown how diversification could reduce risk. In 1973, Fischer Black, Myron Scholes and Robert C. Merton went further by conjuring away risk completely, using the magic trick of dynamic replication. Twenty-five years later, a multi-trillion dollar derivatives industry had grown up around these insights. And of these five founding fathers, only Black missed out on a Nobel prize due to his tragic early death. Black, Scholes and Merton’s option pricing breakthrough depended on the idea that hungry arbitrage traders were constantly prowling the markets, forcing prices to match theoretical predictions. The hedge fund Long-Term Capital Management – which included Scholes and Merton as partners – was founded with this principle at its core. So strong was LTCM’s faith in these theories that it used leverage to make enormous bets on small discrepancies from the predictions of finance theory. We all know what happened next. In August and September 1998, the fund lost $4.5 billion, roughly 90% of its value, and had to be bailed out by its 14 biggest counterparties. Global markets were severely disrupted for several months. All the shibboleths of finance theory, in particular diversification and replication, proved to be false gods, and the reputation of quants suffered badly as a result. Traditionally, finance texts take these shibboleths as a starting point, and build on them. Empirical verification is given scant attention, and the consequences of violating the key assumptions are often ignored completely. The result is a culture where markets get blamed if the theory breaks down, rather than vice versa, as it should be. Unsurprisingly, traders accuse some quants of having an ivory-tower mentality. Now, here come Bouchaud and Potters. Without eschewing rigour, they approach finance theory with a sceptical eye. All the familiar results -- efficient portfolios, Black–Scholes and so on -- are here, but with a strongly empirical flavour. There are also some useful additions to the existing toolkit, such as random matrix theory. Perhaps one day, theorists will show that the exact Black–Scholes regime is an unstable,
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pathological state rather than the utopia it was formerly thought to be. Until then, quants will find this book a useful survival guide in the real world.

Nick Dunbar
Technical Editor, Risk Magazine
Author of Inventing Money (John Wiley and Sons, 2000)
Preface

Finance is a rapidly expanding field of science, with a rather unique link to applications. Correspondingly, recent years have witnessed the growing role of financial engineering in market rooms. The possibility of easily accessing and processing huge quantities of data on financial markets opens the path to new methodologies, where systematic comparison between theories and real data not only becomes possible, but mandatory. This perspective has spurred the interest of the statistical physics community, with the hope that methods and ideas developed in the past decades to deal with complex systems could also be relevant in finance. Correspondingly, many holders of PhDs in physics are now taking jobs in banks or other financial institutions.

However, the existing literature roughly falls into two categories: either rather abstract books from the mathematical finance community, which are very difficult for people trained in natural sciences to read, or more professional books, where the scientific level is usually quite poor. In particular, there is in this context no book discussing the physicists’ way of approaching scientific problems, in particular a systematic comparison between ‘theory’ and ‘experiments’ (i.e. empirical results), the art of approximations and the use of intuition. Moreover, even in excellent books on the subject, such as the one by J. C. Hull, the point of view on derivatives is the traditional one of Black and Scholes, where the whole pricing methodology is based on the construction of riskless strategies. The idea of zero risk is counter-intuitive and the reason for the existence of these riskless strategies in the Black–Scholes theory is buried in the premises of Ito’s stochastic differential rules.

It is our belief that a more intuitive understanding of these theories is needed for a better overall control of financial risks. The models discussed in Theory of

1 There are notable exceptions, such as the remarkable book by J. C. Hull, Futures, Options and Other Derivatives, Prentice Hall, 1997.

Financial Risk are devised to account for real markets’ statistics where the construction of riskless hedges is in general impossible. The mathematical framework required to deal with these cases is however not more complicated, and has the advantage of making the issues at stake, in particular the problem of risk, more transparent.

Finally, commercial software packages are being developed to measure and control financial risks (some following the ideas developed in this book).\textsuperscript{3} We hope that this book can be useful to all people concerned with financial risk control, by discussing at length the advantages and limitations of various statistical models.

Despite our efforts to remain simple, certain sections are still quite technical. We have used a smaller font to develop more advanced ideas, which are not crucial to understanding of the main ideas. Whole sections, marked by a star (*), contain rather specialized material and can be skipped at first reading. We have tried to be as precise as possible, but have sometimes been somewhat sloppy and non-rigorous. For example, the idea of probability is not axiomatized: its intuitive meaning is more than enough for the purpose of this book. The notation $P(\cdot)$ means the probability distribution for the variable which appears between the parentheses, and not a well-determined function of a dummy variable. The notation $x \to \infty$ does not necessarily mean that $x$ tends to infinity in a mathematical sense, but rather that $x$ is large. Instead of trying to derive results which hold true in any circumstances, we often compare order of magnitudes of the different effects: small effects are neglected, or included perturbatively.\textsuperscript{4}

Finally, we have not tried to be comprehensive, and have left out a number of important aspects of theoretical finance. For example, the problem of interest rate derivatives (swaps, caps, swaptions...) is not addressed – we feel that the present models of interest rate dynamics are not satisfactory (see the discussion in Section 2.6). Correspondingly, we have not tried to give an exhaustive list of references, but rather to present our own way of understanding the subject. A certain number of important references are given at the end of each chapter, while more specialized papers are given as footnotes where we have found it necessary.

This book is divided into five chapters. Chapter 1 deals with important results in probability theory (the Central Limit Theorem and its limitations, the theory of extreme value statistics, etc.). The statistical analysis of real data, and the empirical determination of the statistical laws, are discussed in Chapter 2. Chapter 3 is concerned with the definition of risk, value-at-risk, and the theory of optimal

\textsuperscript{3} For example, the software Profiler, commercialized by the company ATSM, heavily relies on the concepts introduced in Chapter 3.

\textsuperscript{4} $a \simeq b$ means that $a$ is of order $b$, $a \ll b$ means that $a$ is smaller than, say, $b/10$. A computation neglecting terms of order $(a/b)^2$ is therefore accurate to 1%. Such a precision is usually enough in the financial context, where the uncertainty on the value of the parameters (such as the average return, the volatility, etc.), is often larger than 1%.
portfolio, in particular in the case where the probability of extreme risks has to be minimized. The problem of forward contracts and options, their optimal hedge and the residual risk is discussed in detail in Chapter 4. Finally, some more advanced topics on options are introduced in Chapter 5 (such as exotic options, or the role of transaction costs). Finally, a short glossary of financial terms, an index and a list of symbols are given at the end of the book, allowing one to find easily where each symbol or word was used and defined for the first time.

This book appeared in its first edition in French, under the title: Théorie des Risques Financiers, Aléa-Saclay-Eyrolles, Paris (1997). Compared to this first edition, the present version has been substantially improved and augmented. For example, we discuss the theory of random matrices and the problem of the interest rate curve, which were absent from the first edition. Furthermore, several points have been corrected or clarified.

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This book is dedicated to our families, and more particularly to the memory of Paul Potters.

Paris, 1999

Jean-Philippe Bouchaud
Marc Potters

With whom we discussed Eq. (1.24), which appears in his Diplomarbeit.