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Excerpt
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1
Preliminaries

In this chapter we collect results from linear algebra and real and complex analysis which we shall use in this book. We will also introduce the definitions and terminology used. Some special functions are also introduced in the present chapter, but the q-series and related material are not defined until Chapter 11. See Chapters 11 and 12 for q-series.

1.1 Hermitian Matrices and Quadratic Forms

Recall that a matrix $A = (a_{j,k})$, $1 \le j, k \le n$ is called Hermitian if

$$\overline{a_{j,k}} = a_{k,j}, \quad 1 \le j, k \le n. \tag{1.1.1}$$

We shall use the following inner product on the n-dimensional complex space $\mathbb{C}^n$,

$$(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} x_j \, \overline{y_j}, \quad \mathbf{x} = (x_1, \dots, x_n)^T, \quad \mathbf{y} = (y_1, \dots, y_n)^T, \tag{1.1.2}$$

where A^T is the transpose of A. Clearly

$$(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}, \qquad (a\mathbf{x}, \mathbf{y}) = a(\mathbf{x}, \mathbf{y}), \quad a \in \mathbb{C}.$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are called orthogonal if $(\mathbf{x}, \mathbf{y}) = 0$. The adjoint A^* of A is the matrix satisfying

$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y}). \tag{1.1.3}$$

It is easy to see that if $A = (a_{j,k})$ then $A^* = (\overline{a_{k,j}})$. Thus, A is Hermitian if and only if $A^* = A$. The eigenvalues of Hermitian matrices are real. This is so since $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \ne 0$ then

$$\lambda(\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, \mathbf{x}) = (\mathbf{x}, A^*\mathbf{x}) = (\mathbf{x}, \lambda\mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x}).$$

Furthermore, the eigenvectors corresponding to distinct eigenvalues are orthogonal. This is the case because if $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{y} = \lambda_2\mathbf{y}$ then

$$\lambda_1(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y}) = \lambda_2(\mathbf{x}, \mathbf{y}),$$

hence $(\mathbf{x}, \mathbf{y}) = 0$.

Any Hermitian matrix generates a quadratic form

$$\sum_{j,k=1}^{n} a_{j,k}\,\overline{x_j}\,x_k, \tag{1.1.4}$$

and conversely any quadratic form with $\overline{a_{j,k}} = a_{k,j}$ determines a Hermitian matrix A through

$$\sum_{j,k=1}^{n} a_{j,k}\,\overline{x_j}\,x_k = \mathbf{x}^* A\mathbf{x} = (A\mathbf{x}, \mathbf{x}). \tag{1.1.5}$$

In an infinite dimensional Hilbert space $\mathcal{H}$, the adjoint is defined by (1.1.3) provided it holds for all $x, y \in \mathcal{H}$. A linear operator A defined in $\mathcal{H}$ is called self-adjoint if $A = A^*$. On the other hand, A is called symmetric if

$$(Ax, y) = (x, Ay)$$

whenever both sides are defined.

Theorem 1.1.1 *Assume that the entries of a matrix A satisfy $|a_{j,k}| \le M$ for all j, k and that each row of A has at most ℓ nonzero entries. Then all the eigenvalues of A satisfy*

$$|\lambda| \le \ell M.$$

Proof Take $\mathbf{x}$ to be an eigenvector of A with an eigenvalue λ, and assume that $\|\mathbf{x}\| = 1$. Observe that the Cauchy–Schwartz inequality implies

$$|\lambda|^2 = |(A\mathbf{x}, \mathbf{x})|^2 = \left|\sum_{j,k=1}^{n} a_{j,k}\,\overline{x_j}\,x_k\right|^2 \le \|x\|^2 \sum_{j=1}^{n}\left|\sum_{k=1}^{n} a_{j,k} x_k\right|^2$$
$$\le \ell^2 M^2.$$

Hence the theorem is proved. □

A quadratic form (1.1.4) is *positive definite* if $(A\mathbf{x}, \mathbf{x}) > 0$ for any nonzero $\mathbf{x}$. Recall that a matrix U is unitary if $U^* = U^{-1}$. The spectral theorem for Hermitian matrices is:

Theorem 1.1.2 *For every Hermitian matrix A there is a unitary matrix U whose columns are the eigenvectors of A such that*

$$A = U^* \Lambda U, \tag{1.1.6}$$

and Λ is the diagonal matrix formed by the corresponding eigenvalues of A.

For a proof see (Horn & Johnson, 1992). An immediate consequence of Theorem 1.1.2 is the following corollary.

Corollary 1.1.3 *The quadratic form* (1.1.4) *is reducible to a sum of squares,*

$$\sum_{j,k=1}^{n} a_{j,k}\, \overline{x_j}\, x_k = \sum_{k=1}^{n} \lambda_k \left|y_k\right|^2, \tag{1.1.7}$$

where $\mathbf{y} = U\mathbf{x}$*, and* $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of* A*.*

The following important characterization of positive definite forms follows from Corollary 1.1.3.

Theorem 1.1.4 *The quadratic form* (1.1.4)–(1.1.5) *is positive definite if and only if the eigenvalues of* A *are positive.*

We next state the Sylvester criterion for positive definiteness (Shilov, 1977), (Horn & Johnson, 1992).

Theorem 1.1.5 *The quadratic form* (1.1.5) *is positive definite if and only if the principal minors of* A*, namely*

$$a_{1,1}, \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}, \dots, \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}, \tag{1.1.8}$$

are positive.

Recall that a matrix $A = (a_{j,k})$ is called **strictly diagonally dominant** if

$$2\left|a_{j,j}\right| > \sum_{k=1}^{n} \left|a_{j,k}\right|. \tag{1.1.9}$$

The following criterion for positive definiteness is in (Horn & Johnson, 1992, Theorem 6.1.10).

Theorem 1.1.6 *Let* A *be* $n \times n$ *matrix which is Hermitian, strictly diagonally dominant, and its diagonal entries are positive. Then* A *is positive definite.*

1.2 Some Real and Complex Analysis

We need some standard results from real and complex analysis which we shall state without proofs and provide references to where proofs can be found. We shall normalize functions of bounded variations to be continuous on the right.

Theorem 1.2.1 (Helly's selection principle) *Let* $\{\psi_n(x)\}$ *be a sequence of uniformly bounded nondecreasing functions. Then there is a subsequence* $\{\psi_{k_n}(x)\}$ *which converges to a nondecreasing bounded function,* ψ*. Moreover if for every* n *the moments* $\int_{\mathbb{R}} x^m\, d\psi_n(x)$ *exist for all* m*,* $m = 0, 1, \dots$*, then the moments of* ψ *exist and* $\int_{\mathbb{R}} x^m\, d\psi_{n_k}(x)$ *converges to* $\int_{\mathbb{R}} x^m\, d\psi(x)$*. Furthermore if* $\{\psi_n(x)\}$ *does not converge, then there are at least two such convergent subsequences.*

For a proof we refer the reader to Section 3 of the introduction to Shohat and Tamarkin (Shohat & Tamarkin, 1950).

Theorem 1.2.2 (Vitali) *Let $\{f_n(z)\}$ be a sequence of functions analytic in a domain $\mathcal{D}$ and assume that $f_n(z) \to f(z)$ pointwise in $\mathcal{D}$. Then $f_n(z) \to f(z)$ uniformly in any subdomain bounded by a contour C, provided that C is contained in $\mathcal{D}$.*

A proof is in Titchmarsh (Titchmarsh, 1964, page 168).

We now briefly discuss the Lagrange inversion and state two useful identities that will be used in later chapters.

Theorem 1.2.3 (Lagrange) *Let $f(z)$ and $\phi(z)$ be functions of z analytic on and inside a contour C containing the point a in its interior. Let t be such that $|t\phi(z)| < |z-a|$ on the contour C. Then the equation*

$$\zeta = a + t\phi(\zeta), \tag{1.2.1}$$

regarded as an equation in ζ, has one root interior to C; and further any function of ζ analytic on the closure of the interior of C can be expanded as a power series in t by the formula

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left[\frac{d^{n-1} f(x)[\phi(x)]^n}{dx^{n-1}}\right]_{x=a}. \tag{1.2.2}$$

See Whittaker and Watson (Whittaker & Watson, 1927, §7.32), or Polya and Szegő (Pólya & Szegő, 1972, p. 145). An equivalent form is

$$\frac{f(\zeta)}{1 - t\phi'(\zeta)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\frac{d^n f(x)[\phi(x)]^n}{dx^n}\right]_{x=a} \tag{1.2.3}$$

Two important special cases are $\phi(z) = e^z$, or $\phi(z) = (1+z)^\beta$. These cases lead to:

$$e^{\alpha z} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+n)^{n-1}}{n!} w^n, \quad w = ze^{-z}, \tag{1.2.4}$$

$$(1+z)^\alpha = 1 + \alpha \sum_{n=1}^{\infty} \binom{\alpha+\beta n-1}{n-1} \frac{w^n}{n}, \quad w = z(1+z)^{-\beta}. \tag{1.2.5}$$

We say that (Olver, 1974)

$$f(z) = O(g(z)), \quad \text{as } z \to a,$$

if $f(z)/g(z)$ is bounded in a neighborhood of $z = a$. On the other hand we write

$$f(z) = o(g(z)), \quad \text{as } z \to a$$

if $f(z)/g(z) \to 0$ as $z \to a$.

A very useful method to determine the large n behavior of orthogonal polynomials $\{p_n(x)\}$ is Darboux's asymptotic method.

Theorem 1.2.4 *Let $f(z)$ and $g(z)$ be analytic in $\{z : |z| < r\}$ and assume that*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad g(z) = \sum_{n=0}^{\infty} g_n z^n, \quad |z| < r. \tag{1.2.6}$$

If $f - g$ is continuous on the closed disc $\{z : |z| \le r\}$ then

$$f_n = g_n + o\left(r^{-n}\right). \tag{1.2.7}$$

This form of Darboux's method is in (Olver, 1974, Ch. 8) and, in view of Cauchy's formulas, is just a restatement of the Riemann–Lebesgue lemma. For a given function f, g is called a comparison function. Another proof of Darboux's lemma is in (Knuth & Wilf, 1989).

In order to apply Darboux's method to a sequence $\{f_n\}$ we need first to find a generating function for the f_n's, that is, find a function whose Taylor series expansion around $z = 0$ has coefficients $c_n f_n$, for some simple sequence $\{c_n\}$. In this work we pay particular attention to generating functions of orthogonal polynomials and Darboux's method will be used to derive asymptotic expansions for some of the orthogonal polynomials treated in this work. The recent work (Wong & Zhao, 2005) shows how Darboux's method can be used to derive uniform asymptotic expansions. This is a major simplification of the version in (Fields, 1967). Wang and Wong developed a discrete version of the Liouville–Green approximation (WKB) in (Wang & Wong, 2005a). This gives uniform asymptotic expansions of a basis of solutions of three-term recurrence relations. This technique is relevant, because all orthogonal polynomials satisfy three-term recurrence relations.

The Perron–Stieltjes inversion formula, see (Stone, 1932, Lemma 5.2), is

$$F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}, \quad z \notin \mathbb{R} \tag{1.2.8}$$

if and only if

$$\mu(t) - \mu(s) = \lim_{\epsilon \to 0^+} \int_s^t \frac{F(x - i\epsilon) - F(x + i\epsilon)}{2\pi i}\, dx. \tag{1.2.9}$$

The above inversion formula enables us to recover μ from knowing its Stieltjes transform $F(z)$.

Remark 1.2.1 *It is clear that if μ has an isolated atom u at $x = a$ then $z = a$ will be a pole of F with residue equal to u. Conversely, the poles of F determine the location of the isolated atoms of μ and the residues determine the corresponding masses. Formula* (1.2.9) *captures this behavior and reproduces the residue at an isolated singularity.*

Remark 1.2.2 *Formula* (1.2.9) *shows that the absolutely continuous component of μ is given by*

$$\mu'(x) = \left[F\left(x - i0^+\right) - F\left(x + i0^+\right)\right]/(2\pi i). \tag{1.2.10}$$

An analytic function defined on a closed disc is bounded and its absolute value attains its maximum on the boundary.

Definition 1.2.1 *Let f be an entire function. The maximum modulus is*

$$M(r; f) := \sup\left\{|f(z)| : |z| \le r\right\}, \quad r > 0. \tag{1.2.11}$$

The order of f, $\rho(f)$ is defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r}. \tag{1.2.12}$$

Theorem 1.2.5 ((Boas, Jr., 1954)) *If $\rho(f)$ is finite and is not equal to a positive integer, then f has infinitely many zeros.*

If f has finite order, its type σ is

$$\sigma = \inf\left\{K : M(r) < \exp\left(K r^{\rho}\right)\right\}. \tag{1.2.13}$$

For an entire function of finite order and type we define the Phragmén–Lindelöf indicator $h(\theta)$ as

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln\left|f\left(re^{i\theta}\right)\right|}{r^{\rho}}. \tag{1.2.14}$$

Consider the infinite product

$$P = \prod_{n=1}^{\infty} (1 + a_n). \tag{1.2.15}$$

We say the P converges to ℓ, $\ell \ne 0$, if

$$\lim_{m\to\infty} \prod_{n=1}^{m} (1 + a_n) = \ell.$$

If $\ell = 0$ we say P diverges to zero. One can prove, see (Rainville, 1960, Chapter 1), that $a_n \to 0$ is necessary for P to converge. Similarly, one can define absolute convergence of infinite products. When $a_n = a_n(z)$ are functions of z, say, we say that P converges uniformly in a domain $\mathcal{D}$ if the partial products

$$\prod_{n=1}^{m} (1 + a_n(z))$$

converge uniformly in $\mathcal{D}$ to a function with no zeros in $\mathcal{D}$.

Definition 1.2.2 *Given a set of distinct points $\{x_j : 1 \le j \le n\}$, the Lagrange fundamental polynomial $\ell_k(x)$ is*

$$\ell_k(x) = \prod_{\substack{j=1\\ j\ne k}}^{n} \frac{(x - x_j)}{(x_k - x_j)} = \frac{S_n(x)}{S_n'(x_k)(x - x_k)}, \quad 1 \le k \le n, \tag{1.2.16}$$

where $S_n(x) = \prod_1^n (x - x_j)$. *The Lagrange interpolation polynomial of a function* $f(x)$ *at the nodes* $x_1, \dots, x_n$ *is the unique polynomial* $L(x)$ *of degree* $n-1$ *such that* $f(x_j) = L(x_j)$.

It is easy to see that $L(x)$ in Definition 1.2.2 is

$$L(x) = \sum_{k=1}^{n} \ell_k(x) f(x_k) = \sum_{k=1}^{n} f(x_k) \frac{S_n(x)}{S_n'(x_k)(x - x_k)}. \tag{1.2.17}$$

Theorem 1.2.6 (Poisson Summation Formula) *Let* $f \in L_1(\mathbb{R})$ *and* F *be its Fourier transform,*

$$F(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-ixt}\, dx, \quad t \in \mathbb{R}.$$

Then

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-inx}\, dx.$$

For a proof, see (Zygmund, 1968, §II.13).

Theorem 1.2.7 *Given two differential equations in the form*

$$\frac{d^2u}{dz^2} + f(z)u(z) = 0, \quad \frac{d^2v}{dz^2} + g(z)v(z) = 0,$$

then $y = uv$ *satisfies*

$$\frac{d}{dz}\left\{\frac{y''' + 2(f+g)y' + (f'+g')\,y}{f-g}\right\} + (f-g)y = 0, \text{ if } f \neq g \tag{1.2.18}$$

$$y''' + 4fy' + 2f'y = 0, \quad \text{if } f = g. \tag{1.2.19}$$

A proof of Theorem 1.2.6 is in Watson (Watson, 1944, §5.4), where he attributes the theorem to P. Appell.

Lemma 1.2.8 *Let* $y = y(x)$ *satisfy the differential equation*

$$\phi(x)y''(x) + y(x) = 0, \quad a < x < b \tag{1.2.20}$$

where $\phi(x) > 0$, *and* $\phi'(x)$ *is positive (negative) and continuous on* (a, b). *Then the successive relative maxima of* $|y|$, *increase (decrease) with* x *in* (a, b) *if* ϕ *increases (decreases) on* (a, b).

Proof Let

$$f(x) := \{y(x)\}^2 + \phi(x)\{y'(x)\}^2, \tag{1.2.21}$$

so that $f(x) = \{y(x)\}^2$ if $y'(x) = 0$. Clearly

$$\begin{aligned} f'(x) &= y'(x)\{2y(x) + \phi'(x)y'(x) + 2\phi(x)y''(x)\} \\ &= \phi'(x)\{y'(x)\}^2. \end{aligned}$$

Thus sign $f'(x) = \text{sign}\,\phi'$ in between the consecutive successive maxima of $|y|$ and the result follows. □

1.3 Some Special Functions

Standard references in this area are (Andrews et al., 1999), (Bailey, 1935), (Rainville, 1960), (Erdélyi et al., 1953b), (Slater, 1964), (Whittaker & Watson, 1927).

The gamma and beta functions are probably the most important functions in mathematics beyond the exponential and logarithmic functions. Recall that

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \text{Re } z > 0, \tag{1.3.1}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \text{Re } x > 0, \quad \text{Re } y > 0. \tag{1.3.2}$$

They are related through

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \tag{1.3.3}$$

The functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{1.3.4}$$

extends the gamma function to a meromorphic function with poles at $z = 0, -1, \dots,$ and also extends $B(x,y)$ to a meromorphic function of x and y. The Mittag–Leffler expansion for Γ'/Γ is (Whittaker & Watson, 1927, §12.3)

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n}\right], \tag{1.3.5}$$

where γ is the Euler constant, (Rainville, 1960, §7).

The shifted factorial is

$$(a)_0 := 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n > 0, \tag{1.3.6}$$

hence (1.3.4) gives

$$(a)_n = \Gamma(a+n)/\Gamma(a). \tag{1.3.7}$$

The shifted factorial is also called Pochhammer symbol. Note that (1.3.7) is meaningful for any complex n, when $a + n$ is not a pole of the gamma function. The gamma function and the shifted factorial satisfy the duplication formulas

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}, \quad (2a)_{2n} = 2^{2n}(a)_n(a+1/2)_n. \tag{1.3.8}$$

We also have the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{1.3.9}$$

We define the multishifted factorial as

$$(a_1, \cdots, a_m)_n = \prod_{j=1}^{m} (a_j)_n\,.$$

Some useful identities are

$$(a)_m(a+m)_n = (a)_{m+n}, \quad (a)_{N-k} = \frac{(a)_N(-1)^k}{(-a-N+1)_k}. \tag{1.3.10}$$

A hypergeometric series is

$$\begin{aligned} {}_rF_s\left(\left.\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right| z\right) &= {}_rF_s\left(a_1,\dots,a_r;b_1,\dots,b_s;z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_s)_n}\frac{z^n}{n!}. \end{aligned} \tag{1.3.11}$$

If one of the numerator parameters is a negative integer, say $-k$, then the series (1.3.11) becomes a finite sum, $0 \le n \le k$ and the ${}_rF_s$ series is called *terminating*. As a function of z nonterminating series is entire if $r \le s$, is analytic in the unit disc if $r = s+1$. The hypergeometric function ${}_2F_1(a,b;c;z)$ satisfies the hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0. \tag{1.3.12}$$

The confluent hypergeometric function (Erdélyi et al., 1953b, §6.1)

$$\Phi(a,c;z) := {}_1F_1(a;c;z) \tag{1.3.13}$$

satisfies the differential equation

$$z\frac{d^2y}{dz^2} + (c-z)\frac{dy}{dz} - ay = 0, \tag{1.3.14}$$

and $\lim\limits_{b\to\infty} {}_2F_1(a,b;cz/b) = {}_1F_1(a;c;z)$. The Tricomi Ψ function is a second linear independent solution of (1.3.14) and is defined by (Erdélyi et al., 1953b, §6.5)

$$\Psi(a,c;x) := \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c;x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a-c+1,2-c;x). \tag{1.3.15}$$

The function of Ψ has the integral presentation (Erdélyi et al., 1953a, §6.5)

$$\Psi(a,c;x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{c-a-1}\,dt, \tag{1.3.16}$$

for $\operatorname{Re} a > 0$, $\operatorname{Re} x > 0$.

The Bessel function J_ν and the modified Bessel function I_ν, (Watson, 1944) are

$$\begin{aligned} J_\nu(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{\Gamma(n+\nu+1)\,n!}, \\ I_\nu(z) &= \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{\Gamma(n+\nu+1)\,n!}. \end{aligned} \tag{1.3.17}$$

Clearly $I_\nu(z) = e^{-i\pi\nu/2}J_\nu\left(ze^{i\pi/2}\right)$. Furthermore

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z. \tag{1.3.18}$$

The Bessel functions satisfy the recurrence relation

$$\frac{2\nu}{z} J_\nu(z) = J_{\nu+1}(z) + J_{\nu-1}(z). \tag{1.3.19}$$

The Bessel functions J_ν and $J_{-\nu}$ satisfy

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right) y = 0. \tag{1.3.20}$$

When ν is not an integer J_ν and $J_{-\nu}$ are linear independent solutions of (1.3.20) whose Wronskian is (Watson, 1944, §3.12)

$$W\{J_\nu(x), J_{-\nu}(x)\} = -\frac{2\sin(\nu\pi)}{\pi x}, \quad W\{f, g\} := fg' - gf'. \tag{1.3.21}$$

The function I_ν satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2 + \nu^2\right) y = 0, \tag{1.3.22}$$

whose second solution is

$$\begin{aligned} K_\nu(x) &= \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}, \\ K_n(x) &= \lim_{\nu\to n} K_\nu(x), \quad n = 0, \pm 1, \dots . \end{aligned} \tag{1.3.23}$$

We also have

$$\begin{aligned} I_{\nu-1}(x) - I_{\nu+1}(x) &= \frac{2\nu}{x} I_\nu(x), \\ K_{\nu+1}(x) - K_{\nu-1}(x) &= \frac{2\nu}{x} K_\nu(x). \end{aligned} \tag{1.3.24}$$

Theorem 1.3.1 *When $\nu > -1$, the function $z^{-\nu} J_\nu(z)$ has only real and simple zeros. Furthermore, the positive (negative) zeros of $J_\nu(z)$ and $J_{\nu+1}(z)$ interlace for $\nu > -1$.*

We shall denote the positive zeros of $J_\nu(z)$ by $\{j_{\nu,k}\}$, that is

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots . \tag{1.3.25}$$

The Bessel functions satisfy the differential recurrence relations, (Watson, 1944)

$$zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z), \tag{1.3.26}$$

$$zY'_\nu(z) = \nu Y_\nu(z) - zY_{\nu+1}(z), \tag{1.3.27}$$

$$zI_\nu(z) = zI_{\nu+1}(z) + \nu I_\nu(z), \tag{1.3.28}$$

$$zK'_\nu(z) = \nu K_\nu(z) - zK_{\nu+1}(z), \tag{1.3.29}$$

where $Y_\nu(z)$ is

$$\begin{aligned} Y_\nu(z) &= \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \quad \nu \neq 0, \pm 1, \dots, \\ Y_n(z) &= \lim_{\nu\to n} Y_\nu(z), \quad n = 0, \pm 1, \dots . \end{aligned} \tag{1.3.30}$$

The functions $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent solutions of (1.3.20).