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Sections or chapters marked with a dagger (†) can be skipped on a first reading of the book. They contain material that is not central to the book, or they include more advanced or more technical subject matter. However, they also sometimes address topics that are at the forefront of current research.

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