Some of the greatest scientists, including Poisson, Faraday, Maxwell, Rayleigh, and Einstein, have contributed to the theory of composite materials. Mathematically, it is the study of partial differential equations with rapid oscillations in their coefficients. Although extensively studied for more than 100 years, an explosion of ideas in the last four decades (and particularly in the last two decades) has dramatically increased our understanding of the relationship between the properties of the constituent materials, the underlying microstructure of a composite, and the overall effective (electrical, thermal, elastic) moduli that govern the macroscopic behavior. This renaissance has been fueled by the technological need for improving our knowledge base of composites, by the advance of the underlying mathematical theory of homogenization, by the discovery of new variational principles, by the recognition of how important the subject is to solving structural optimization problems, and by the realization of the connection with the mathematical problem of quasiconvexification. This book surveys these exciting developments at the frontier of mathematics and presents many new results.

Graeme W. Milton is a Distinguished Professor in the Mathematics Department at the University of Utah. He has been awarded Sloan and Packard Fellowships and is on the editorial board of the *Archive for Rational Mechanics and Analysis.* He has published more than 70 papers on the theory of composite materials.

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Cover drawing of a composite material with a negative Poisson’s ratio, used by permission from
G. W. Milton and A. V. Cherkaev, “Which elasticity tensors are realizable?” *ASME Journal of Engineering Materials and Technology* **117**: 483–93 (1995).

To John, Winsome, and John

Contents

Sections or chapters marked with a dagger (†) can be skipped on a first reading of the book. They contain material that is not central to the book, or they include more advanced or more technical subject matter. However, they also sometimes address topics that are at the forefront of current research.

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Preface

This book is intended to be a self-contained introduction to the theory of composite materials, encompassing the electrical, thermal, magnetic, thermoelectric, mechanical, piezoelectric, poroelastic, and electromagnetic properties. It is intended not only for mathematicians, but also for physicists, geophysicists, material scientists, and electrical and mechanical engineers. Consequently, the results are not stated in the format of lemmas, propositions, and theorems. Instead, the focus is on explaining the central ideas and providing proofs that avoid unnecessary technicalities. The book is suitable as a textbook in an advanced-level graduate course, and also as a reference book for researchers working on composites or in related areas.

The field of composite materials is enormous. That's good, because it means that there are many avenues of research to explore. The drawback is that a single book cannot adequately cover the whole field. The main focus of this book is on the relation between the microstructure of composites and the effective moduli that govern their behavior. This choice reflects my research interests, and is also the starting point for many other avenues of research on composites. Topics not treated here include fatigue, fracture, and plastic yielding in composites, which are major factors in determining their strength (Sih and Tamuzs 1979; Sih and Chen 1981; Sih and Skudra 1985; Talreja 1994; Hull and Clyne 1996; Nemat-Nasser and Hori 1999); the propagation, localization, and scattering of waves in composites at wavelengths comparable to or smaller than the size of the inhomogeneities (Sheng 1990, 1995; Chew 1995) [of particular recent interest is the study of photonic band gap materials (Joannopoulos, Meade, and Winn 1995), which may lead to the development of new lasers and could be important in photonic circuitry]; flow in porous media, which has obvious applications to the management of oil and water reservoirs and to understanding the seepage of waste fluids (Scheidegger 1974; Sanchez-Palencia 1980); geometrical questions such as the microstructures of rocks (Pittman 1984) and dense random packings of hard spheres (Cargill III 1984; Torquato, Truskett, and Debenedetti 2000); and the many aspects of percolation theory (Kesten 1982; Stauffer and Aharony 1992; Grimmett 1999).

Other important topics, such as homogenization theory (discussed in chapter 1 on page 1), numerical methods for solving for the fields in composites, and hence for determining their effective moduli (discussed in section 2.8 on page 38), the nonlinear theory of composites (discussed in section 13.7 on page 282), structural optimization (discussed in section 21.3 on page 429), and quasiconvexification (discussed in chapter 31 on page 671) are not treated in the depth that they deserve. The reader is encouraged to refer to the references cited in those sections to gain a more complete understanding of these subjects.

The Contents gives a good indication of what topics the book covers. Briefly, the first chapter discusses the motivation for studying composites and outlines homogenization theory from various viewpoints. The second chapter introduces some of the different equations considered in the book, and numerical methods for solving these equations are mentioned. Chapters 3 to 9 cover exact results for effective moduli, relations between (seemingly unconnected) effective moduli and microstructures for which at least some of the effective moduli can be exactly determined (such as coated sphere assemblages, laminates, and their generalizations). Chapter 10 discusses some of the many approximations that have been developed for estimating effective moduli and the asymptotic formulas that are valid in certain high-contrast materials. Chapter 11 shows how wave propagation in composites, at wavelengths much larger than the microstructure, can be treated by allowing the moduli, fields, and effective moduli to be complex, or alternatively by keeping everything real and doubling the size of the system of equations being considered.

Chapters 12 to 18 cover the general theory concerning effective tensors: the formulation as a problem in Hilbert space; various variational principles; convergent series expansions for the effective tensor in powers of the variation in the local tensor field; how (for random composites) the terms in the series expansion can be expressed in terms of correlation functions; other perturbation solutions for the effective tensor; the general theory of exact relations in composites; and, finally, the analytic properties of the effective tensor as a function of the tensors of the constituent tensors. These chapters (due to their generality) are harder to read than those in the first part of the book. The first part of chapter 12 is essential reading since it introduces some of the basic notation used in subsequent chapters. Also, chapter 13, on variational principles, should certainly be read, and will strengthen the reader's understanding of the material in chapter 12. Chapters 19 and 20 are optional. They introduce the Y-tensor, which in a multicomponent composite gives information about the average fields in each phase, and which in electrical circuits determines the response of the circuit. The theory of Y-tensors parallels that of effective tensors, and many bounds on effective tensors take a simpler form when expressed in terms of the Y-tensor.

Chapter 21 introduces the problem of bounding effective tensors and discusses its importance in optimal design problems. Chapters 22 to 26 describe variational methods for bounding effective tensors, including the Hashin-Shtrikman approach, the translation method (or compensated compactness) approach, and those approaches based on classical variational principles. Chapters 27 and 28 show how the analyticity properties of the effective tensor lead to large families of bounds, which usually are the simplest rational approximants of the function compatible with what is known about it. Chapter 29 outlines the parallel between operations on analytic functions and operations on subspace collections, and shows how this leads to bounds for multicomponent composites.

Chapter 30 discusses general properties and characterizations of the set of effective tensors obtained as the microstructure is varied over all configurations. The set of elastic tensors that can be made by mixing a sufficiently compliant isotropic material with a sufficiently stiff isotropic material is shown to coincide with the set of all positive-definite fourth-order tensors satisfying the symmetries of elasticity tensors. Chapter 31 shows how problems of bounding effective tensors are equivalent to quasiconvexification problems, and vice versa. Finally, by extending a famous example of Šverák, an example is given of a seven-phase composite whose effective elastic tensor cannot be mimicked by any (multiple-rank) laminate material.

There are many other related books that present the theory of composites from other perspectives. Those that are closest in their scope include the following. The report of Hashin (1972), the classic book of Christensen (1979), the books of Agarwal and Broutman (1990), Matthews and Rawlings (1994), and Hull and Clyne (1996), and the recent book of Nemat-Nasser and Hori (1999) cover the subject with an emphasis on the mechanical properties of composites. The book of Zhikov, Kozlov, and Oleinik (1994) covers the subject from a rigorous mathematical perspective. The volume edited by Cherkaev and Kohn (1997) contains translations of many significant mathematical papers, which previously were only available in French or Russian. The books of Allaire (2001) and Cherkaev (2000) cover the subject with an emphasis on structural optimization. The book of Ball and James (2001) surveys many problems where microstructure plays an influential role in determining macroscopic behavior. The book of Beran (1968) covers the statistical theory, using an approach that is different from the one presented in chapter 15 on page 313. The book of Torquato (2001) covers many topics with an emphasis on the statistical aspects of composites. There are also many review papers, including Willis (1981), Hashin (1983), Torquato (1991), Bergman and Stroud (1992), and Markov (2000). Additionally, there are many books on homogenization theory and on quasiconvexification, which are referenced in chapters 1 on page 1 and 31 on page 671.

It is a great pleasure to thank those colleagues and friends who contributed in many ways to this book. I would like to thank Ross McPhedran, who introduced me to the subject of composite materials when I was an undergraduate at Sydney University. I am greatly indebted to Michael Fisher for his critical comments during my Ph.D., which have had a lasting impact. I am grateful to George Papanicolaou for encouraging me to write this book. When I started writing, more than 13 years ago, it was just meant to be one-third of a book and certainly was not intended to be more than 700 pages in length. But I found it difficult to resist the temptation to include topics that seemed to tie in closely with what I had already written, and to include new developments such as novel families of neutral inclusions and the associated exactly solvable assemblages (section 7.11 on page 134), the theory of partial differential laminates (section 9.10 on page 177), the general theory of exact relations in composites (chapter 17 on page 355), the optimal microstructures of Sigmund attaining the Hashin-Shtrikman bounds (section 23.9 on page 481), an approach for finding suitable quasiconvex functions for obtaining bounds (section 25.7 on page 544), and a composite with an effective tensor that cannot be mimicked by laminates (section 31.9 on page 690). John Willis and François Murat are especially thanked for their help in arranging my visits to the University of Bath, and to the Université Paris VI, where major portions of the text were written, and where (in Paris) the counterexample of 31.9 on page 690 was discovered. I am grateful to numerous people for their constructive comments on sections of the text, including Leonid Berlyand, Andrei Cherkaev, Gilles Francfort, Ken Golden, Zvi Hashin, Robert Kohn, Mordehai Milgrom, Vincenzo Nesi, Sergey Serkov, and Luc Tartar. I am thankful to Eleen Collins for typing most of the references into BIBTEX. I am most indebted to Nelson Beebe for the absolutely terrific job he did in developing the software for the book style and referencing style, for automating the conversion of references to BIBTEX, for solving many technical problems, and for spotting many errors. I am also grateful to Thilagavathi Murugesan for her substantial help in checking most of the equations, to Sergei Serkov for scanning many of the figures, and to Elise Oranges for the great copyediting job she did. Additionally, I wish to thank Bob Kohn for suggesting Cambridge University Press, and David Tranah and Alan Harvey at Cambridge University Press for their continued interest and helpful suggestions. I am grateful to my partner, John Patton, and my parents, John and Winsome Milton, for their continued support throughout the whole work. It is a pleasure to dedicate this book to them.

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While I hope that the derivations in the book are correct, and that work has been properly referenced, it is inevitable that there are still some errors and omissions. I would be grateful to learn about these. The Web site `http://www.math.utah.edu/books/tcbook` contains a list of known errors in the book, as well as the BIBTEX bibliographic database.

Salt Lake City, Utah
October, 2001

Graeme W. Milton

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