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Frontmatter
More information

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Lie's Structural Approach to PDE Systems

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

4 W. Miller, Jr. *Symmetry and separation of variables*
6 H. Minc *Permanents*
11 W. B. Jones and W. J. Thron *Continued fractions*
12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter and S. D. Riemenschneider *Birkhoff interpolation*
21 W. T. Tutte *Graph theory*
22 J. R. Bastida *Field extensions and Galois theory*
23 J. R. Cannon *The one-dimensional heat equation*
25 A. Salomaa *Computation and automata*
26 N. White (ed.) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 N. White (ed.) *Combinatorial geometrics*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations containing several variables*
32 M. Kuczma, B. Chozewski and R. Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen and O. Staffans *Volterra integral and functional equations*
35 G. Gasper and M. Rahman *Basic hypergeometric series*
36 E. Torgersen *Comparison of statistical experiments*
37 A. Neumaier *Intervals methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
40 N. White (ed.) *Matroid applications*
41 S. Sakai *Operator algebras in dynamical systems*
42 W. Hodges *Model theory*
43 H. Stahl and V. Totik *General orthogonal polynomials*
44 R. Schneider *Convex bodies*
45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
46 A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented matroids*
47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
48 C. Sims *Computation with finitely presented groups*
49 T. Palmer *Banach algebras and the general theory of *-algebras*
50 F. Borceux *Handbook of categorical algebra I*
51 F. Borceux *Handbook of categorical algebra II*
52 F. Borceux *Handbook of categorical algebra III*
54 A. Katok and B. Hassleblatt *Introduction to the modern theory of dynamical systems*
55 V. N. Sachkov *Combinatorial methods in discrete mathematics*
56 V. N. Sachkov *Probabilistic methods in discrete mathematics*
57 P. M. Cohn *Skew Fields*
58 Richard J. Gardner *Geometric tomography*
59 George A. Baker, Jr. and Peter Graves-Morris *Padé approximants*
60 Jan Krajicek *Bounded arithmetic, propositional logic, and complex theory*
61 H. Gromer *Geometric applications of Fourier series and spherical harmonics*
62 H. O. Fattorini *Infinite dimensional optimization and control theory*
63 A. C. Thompson *Minkowski geometry*
64 R. B. Bapat and T. E. S. Raghavan *Nonnegative matrices and applications*
65 K. Engel *Sperner theory*
66 D. Cvetkovic, P. Rowlinson and S. Simic *Eigenspaces of graphs*
67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial species and tree-like structures*
68 R. Goodman and N. Wallach *Representations of the classical groups*
69 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume I 2 ed.*
70 A. Pietsch and J. Wenzel *Orthonormal systems and Banach space geometry*
71 George E. Andrews, Richard Askey and Ranjan Roy *Special Functions*
72 R. Ticciati *Quantum field theory for mathematicians*
76 A. A. Ivanov *Geometry of sporadic groups I*
78 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume II 2 ed.*

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Lie's Structural Approach to PDE Systems

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Contents

Preface

Das Ziel der Wissenschaft ist einerseits, neue Tatsachen zu erobern, anderseits, bekannte unter höheren Gesichtspunkten zusammenfassen.

S. Lie

One of the principal objects of theoretical research is to find the point of view from which the subject appears in its greatest simplicity.

J.W. Gibbs

Good mathematics = down-to-earth mathematics.

A. Andreotti

Everyone knows that the theory of PDE systems is enormously rich in results—but *what about foundations*? This monograph describes one approach, but there is no claim that it is the only one, or that it is the best possible; just that there is one, and moreover one that has been around for a long time without having been as recognized as it deserves.

The study is restricted to local solvability. If a PDE system $\mathcal{S}$ is defined in a domain $\mathcal{D}$, and if it can be shown that $\mathcal{S}$ possesses local solutions at each point of $\mathcal{D}$, the question of global solvability boils down to whether it is possible to glue together local solutions in order to form global ones—in analogy with the cohomology theory for coherent analytic sheaves. But *first of all one has to know that there are local solutions.*

A major idea is to regard PDE theory from the point of view of *differential geometry*—rather than basing it on analysis, say.

To illustrate this, let us first find a suitable angle from which ODE systems appear in a simple way.

Example. Look for functions $y(t)$, $z(t)$ satisfying the ODE system

$$\begin{cases} y'' = f(t, y, z, y', z'), \\ z'' = g(t, y, z, y', z'). \end{cases}$$

With $x^1 := y$, $x^2 := z$, $x^3 := y'$ and $x^4 := z'$, this goes over into the first order system

$$\begin{cases} dx^1/dt = x^3, \\ dx^2/dt = x^4, \\ dx^3/dt = f(t, x^1, x^2, x^3, x^4), \\ dx^4/dt = g(t, x^1, x^2, x^3, x^4). \end{cases}$$

Therefore the key problem in ODE theory consists in solving first order systems

$$\begin{cases} dx^1/dt = f^1(t, x^1, x^2, \dots, x^n), \\ \dots \\ dx^n/dt = f^n(t, x^1, x^2, \dots, x^n). \end{cases} \qquad (*)$$

The classical existence and uniqueness theorem says that for smooth f^k there is precisely one local solution $(x^1(t), \dots, x^n(t))$ satisfying initial data $(x^1(t_0), \dots, x^n(t_0)) = (x_0^1, \dots, x_0^n)$.

In order to make this conceptually clearer, rewrite (*) as a pfaffian system:

$$\begin{cases} dx^1 - f^1(t, x)\, dt = 0, \\ \dots \\ dx^n - f^n(t, x)\, dt = 0, \end{cases}$$

where $x = (x^1, \dots, x^n)$. Setting

$$\theta^k := dx^k - f^k(t, x)\, dt \quad \text{for} \quad k = 1, \dots, n,$$

a solution $(x^1(t), \dots, x^n(t))$ is a function whose graph

$$\mathbb{R}_t \ni t \mapsto (t, x^1(t), \dots, x^n(t)) \in \mathbb{R}_t \times \mathbb{R}_x^n$$

is an integral curve of the pfaffian system $\theta^1 = \cdots = \theta^n = 0$ on $\mathbb{R}_t \times \mathbb{R}_x^n$.

The dual of the latter consists of those vector fields

$$V = a\frac{\partial}{\partial t} + \sum_{k=1}^{n} a^k \frac{\partial}{\partial x^k}$$

which satisfy $0 = \theta^m(V) = a^m - a \cdot f^m(t,x)$ for $m = 1,\ldots,n$. Consequently they are all multiples of the vector field

$$X = \frac{\partial}{\partial t} + \sum_{k=1}^{n} f^k(t,x)\frac{\partial}{\partial x^k}.$$

Then $(x^1(t),\ldots,x^n(t))$ is a solution of (*) if and only if its graph is an integral curve of X.

Thus solving a first order ODE system is equivalent to finding the integral curves of a vector field. The latter may be done locally by introducing new coordinates making the vector field maximally simple.

The local rectification lemma. *Let*

$$X = \frac{\partial}{\partial t} + \sum_{k=1}^{n} f^k(t,x)\frac{\partial}{\partial x^k}$$

be a vector field defined near the origin of $\mathbb{R}_t \times \mathbb{R}^n_x$. *Then there is a local diffeomorphism*

$$\phi : \mathbb{R}_s \times \mathbb{R}^n_y \xrightarrow{\cong} \mathbb{R}_t \times \mathbb{R}^n_x$$

with $\phi(0) = 0$ *and* $\phi_*(\partial/\partial s) = X$.

This means that the integral curves of X are given by

$$y^k(t,x) = \text{constant} \qquad \text{for } k = 1,\ldots,n.$$

From a technical point of view this theorem is equivalent to the classical local existence theorem for first order ODE systems—but the advantage is that it is much more appealing to the intuition (if you agree with this, read on; otherwise stop right here).

Conclusion. *Solving an ODE system is equivalent to rectifying a vector field—which always can be done locally in the smooth category.*

Since this result is most satisfying, it is natural to ask if something similar works for general PDE systems.

Question. *Is it possible to geometrize PDE systems in an analogous way?*

The first step consists in rewriting an arbitrary PDE system as a pfaffian system, or—dually—a vector field system.

To get the idea, consider a first order ODE:

$$F(x, y, y') = 0.$$

The graph of a solution is given by

$$x \mapsto (x, y(x)).$$

Analogously the 1-graph is given by

$$x \mapsto (x, y(x), y'(x)),$$

the 2-graph by

$$x \mapsto (x, y(x), y'(x), y''(x)),$$

and so on. Let us concentrate on the 1-graph, which is a curve in a 3-dimensional space $J^1(\mathbb{R}_x, \mathbb{R}_y)$ with coordinates x, y and p, say:

$$\begin{cases} x = x, \\ y = y(x), \\ p = y'(x). \end{cases}$$

Note that on this 1-graph

$$dy - p\,dx = 0.$$

The one-form $\theta^0 = dy - p\,dx$ appearing here is called the *contact form*.

Conversely, let c be a curve in the (x, y, p)-space on which x can be used as a local coordinate, and suppose that θ^0 vanishes on c. Then c is of the form

$$x \mapsto (x, f(x), g(x))$$

with

$$0 = df - g\,dx = (f'(x) - g(x))\,dx, \qquad \text{i.e.,} \qquad g(x) = f'(x).$$

But this means that c is the 1-graph of the function $f(x)$.

To the ODE $F(x, y, y') = 0$ corresponds the hypersurface $\boldsymbol{M}$ in the (x, y, p)-space defined by

$$F(x, y, p) = 0.$$

Clearly solutions of $F(x, y, y') = 0$ will correspond to 1-graphs contained in $\boldsymbol{M}$, that is, to curves in $\boldsymbol{M}$ satisfying the two requirements

(i) x can be used as a local coordinate on the curve,
(ii) $\theta^0 \equiv dy - p\,dx = 0$ vanishes on the curve.

Let us assume $\boldsymbol{M}$ to be smooth, and set

$$\theta := \theta^0{}_{|\boldsymbol{M}} = (dy - p\,dx)_{|\boldsymbol{M}},$$

so that θ is a one-form on $\boldsymbol{M}$. Then the solutions of $F(x, y, y') = 0$ correspond precisely to the integral curves of θ on which $dx \neq 0$. If the latter condition is not fulfilled, the integral curve is said to represent a *generalized solution* of the ODE.

Conclusion. *The ODE $F(x, y, y') = 0$ can be regarded as a manifold $\boldsymbol{M}$ equipped with a one-form θ. The solutions of the ODE correspond to integral curves of θ.*

Let us now play the same game for a q^{th} order PDE system $\mathcal{S}$ in n independent variables $x^1, \ldots, x^n$ and m dependent variables $z^1, \ldots, z^m$:

$$\mathcal{S}: \quad F^a\left(x^1, \ldots, x^n; z^1, \ldots, z^m; \ldots, \frac{\partial^k z^j}{\partial x^{i_1} \ldots \partial x^{i_k}}, \ldots\right) = 0,$$

with $a = 1, \ldots, A$ and $k \leq q$. To this set-up corresponds the jet space $J^q(\mathbb{R}^n_x, \mathbb{R}^m_z)$ with coordinates

$$x^1, \ldots, x^n, z^1, \ldots, z^m, \ldots, p^j_{i_1 \ldots i_k}, \ldots,$$

where $p^j_{i_1 \ldots i_k}$ is associated to the derivative

$$\frac{\partial^k z^j}{\partial x^{i_1} \ldots \partial x^{i_k}}.$$

$\mathcal{S}$ corresponds to the subset

$$\boldsymbol{M}: \quad F^a(x^1, \ldots x^n; z^1, \ldots, z^m; \ldots, p^k_{i_1 \ldots i_k}, \ldots) = 0$$

of $J^q(\mathbb{R}^n_x, \mathbb{R}^m_z)$; let us assume $\boldsymbol{M}$ to be smooth.

An n-dimensional submanifold $\mathcal{N}$ of $J^q(\mathbb{R}^n_x, \mathbb{R}^m_z)$ is a q-graph if and only if

(i) $dx^1 \wedge \cdots \wedge dx^n \neq 0$ on $\mathcal{N}$,
(ii) $\mathcal{N}$ is an integral manifold of the contact ideal ${}^qCt^{n,m}$ generated by the one-forms

$$\begin{cases} dz^j - \sum_{i_1=1}^n p^j_{i_1}\, dx^{i_1}, \\ dp^j_{i_1} - \sum_{i_2=1}^n p^j_{i_1 i_2}\, dx^{i_2}, \\ \ldots \\ dp^j_{i_1 \ldots i_{q-1}} - \sum_{i_q=1}^n p^j_{i_1 i_2 \ldots i_q}\, dx^{i_q}. \end{cases}$$

Let $\mathcal{P}$ denote the restriction of ${}^qCt^{n,m}$ to $\boldsymbol{M}$. Then *the set of solutions of $\mathcal{S}$ correspond precisely to the set of n-dimensional integral manifolds of $\mathcal{P}$ on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$.*

Define the dual vector field system $\mathcal{V} = \mathcal{P}^\perp$ on $\boldsymbol{M}$ by

$$X \in \mathcal{V} \Longleftrightarrow \theta(X) = 0 \text{ for all } \theta \in \mathcal{P}.$$

Then

solving the PDE system $\mathcal{S}$ is equivalent to finding all n-dimensional integral manifolds of $\mathcal{V}$ on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$.

Again, suspending the last condition gives *generalized solutions* of $\mathcal{S}$.

The classical terminology for finding integral manifolds of a pfaffian or vector field system is to *integrate* the system.

Conclusion. *Solving a PDE system is equivalent to integrating a pfaffian system or its dual vector field system.*

Consider the vector field version: let $\mathcal{V} = (X_1, \dots X_q)$ be the vector field system generated by the vector fields $X_1, \dots, X_q$ over the ring of smooth functions on the manifold $\boldsymbol{M}$. Shrinking $\boldsymbol{M}$ if necessary, the X_i can be assumed to be linearly independent everywhere. The *derived system* $\mathcal{V}'$ is generated by the X_i and their Lie brackets $[X_i, X_j]$; say that $\mathcal{V}' = (X_1, \dots, X_q; Z_1, \dots, Z_p)$. Then the *structure equations* of $\mathcal{V}$ are given by

$$[X_i, X_j] \equiv \sum_{k=1}^{p} c_{ij}^k Z_k \pmod{\mathcal{V}} \qquad \text{for } i, j = 1, \dots r,$$

with certain *structure functions* c_{ij}^k. The latter depend on the basis of $\mathcal{V}$, and it is natural to choose one that kills as many c_{ij}^k as possible. The resulting set is called the *Lie structure* of $\mathcal{V}$, and the general claim is

the integrability properties of $\mathcal{V}$ are governed by the Lie structure!

One reason for this is Cartan's local existence theorem for integral manifolds. The proof consists of two parts: first linear algebra applied to the structure equations gives all *involutions*, and then these are specialized to *integral manifolds* by repeated applications of the Cauchy–Kowalewski theorem. Unfortunately the latter requires power series, and is not valid in the C^∞ category.

Questions:

(i) *What is required in order to obtain integral manifolds without using power series?*

(ii) *Is there a natural intrinsic property from which more information than just existence can be derived?*

In order to tackle these it is convenient to use the classification of Drach: any PDE system is equivalent to either a first or a second order PDE system in one dependent variable.

First order systems in one dependent variable are easily understood by means of Lie's structural methods, so the real difficulties start with second order systems.

The main part of the monograph is devoted to the study of second order PDE systems in one dependent and two or three independent variables. When considering these, one is naturally led to the notion of *Monge characteristic subsystems*, which in their turn provide the following partial answer to the questions above:

> *If a vector field system admits Monge systems with enough first integrals, then it is possible to find integral manifolds without using the Cauchy–Kowalewski theorem. Moreover these Monge systems yield a lot of interesting information beyond that of pure existence—in particular as regards classifications.*

A big surprise is that looking at second order PDE systems in one dependent variable from this angle, the theory will be dominated by local Lie groups and Lie pseudogroups.

Extending these methods to the case of more than three independent variables would be quite complicated, but surely not impossible. And anyway, who would expect the general PDE theory to be simple?

Most topics treated here can be found in the classical works of Lie, Cartan and Vessiot. However, a great effort has been made to present the ideas in a *unified* and very *simple* manner. In order not to obscure the fundamental issues it has been left to the interested reader to fill in the details needed to obtain his or her own desired level of rigour.