1

Introduction and summary

This is the story of a geometric approach to the theory of PDE systems, initiated by Sophus Lie and developed by two of his disciples, Élie Cartan and Ernest Vessiot.

In chapter **2** it is explained how any decent PDE system $\mathcal{S}$ can be considered as a submanifold of an appropriate jet bundle. The latter is equipped with its canonical contact pfaffian system, the restriction of which to $\mathcal{S}$ makes $\mathcal{S}$ a *manifold with a pfaffian system* $\mathcal{P}$ or, dually, a *manifold with a vector field system* $\mathcal{V}$. The problem of solving the PDE system $\mathcal{S}$ then goes over into that of finding *integral manifolds* of $\mathcal{P}$ (or $\mathcal{V}$) of a prescribed dimension.

Of the three heroes of our tale, Lie and Vessiot favoured the vector field approch, while Cartan is the great champion of differential forms. For our purposes it is important to be able to use both approaches and to have a complete duality, so that each concept for vector field systems has a counterpart for pfaffian systems, and vice versa.

As we are only interested in local properties, $\mathcal{S}$ is assumed to be a small open subset of $\mathbb{R}^r$ (or $\mathbb{C}^r$), and $\mathcal{V}$ is supposed to be generated by vector fields $X_1, \dots, X_q$, independent everywhere on $\mathcal{S}$: $\mathcal{V} = (X_1, \dots, X_q)$.

The simplest case occurs when $\mathcal{V}$ is complete with respect to Lie brackets, that is,

$$[X_i, X_j] \equiv 0 \pmod{\mathcal{V}} \quad \text{for } i, j = 1, \dots, r.$$

According to the Frobenius theorem it is in this case possible to introduce local coordinates $x^1, \dots, x^r$ such that $\mathcal{V} = (\partial/\partial x^1, \dots, \partial/\partial x^q)$, whence the integral manifolds are given by

$$x^{q+1} = \text{constant}, \dots, x^r = \text{constant}.$$

With the derived system $\mathcal{V}'$ being generated by the X_i and their Lie brackets $[X_i, X_j]$, the Frobenius condition can equivalently be written as $\mathcal{V}' = \mathcal{V}$.

The general case with $\mathcal{V}' \supsetneqq \mathcal{V}$ is solved in chapter **3** by means of Cartan's local existence theorem. The key idea is to first look for *maximal involutions*—with an involution being a subsystem $\mathcal{I}$ of the vector field system $\mathcal{V}$ satifying $[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{V}$. Then these involutions are specialized to *complete subsystems*, i.e., subsystems $\mathcal{W}$ of $\mathcal{V}$ with $[\mathcal{W}, \mathcal{W}] \subseteq \mathcal{W}$. Thereupon the Frobenius theorem yields the wanted integral manifolds.

The step from involutions to complete subsystems is based upon a repeated application of the Cauchy–Kowalewski theorem, which unfortunately requires analyticity.

Often one wants n-dimensional integral manifolds on which $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$, where the ω^i are given one-forms. If the general n-dimensional involution $\mathcal{I}_n$ satisfies $\omega^1 \wedge \cdots \wedge \omega^n|_{\mathcal{I}_n} \neq 0$, $\mathcal{V}$ is said to be involutive with respect to the ω^i. In this case Cartan's procedure yields integral manifolds of the kind wanted.

Any vector field system $\mathcal{V}$ can be *prolonged* to a vector field system $\mathcal{V}^{(1)}$ on a higher dimensional manifold, and this in turn can be prolonged to $\mathcal{V}^{(2)}$, and so on. Moreover there is a one-to-one correspondence between the integral manifolds of $\mathcal{V}$ and those of $\mathcal{V}^{(k)}$ for $k = 1, 2, 3, \ldots$.

Chapter **4** sketches the prolongation theorem of Cartan and Janet, which says that by a *finite* number of prolongations it is possible to conclude either that some $\mathcal{V}^{(m)}$ is involutive with respect to $\omega^1 \wedge \cdots \wedge \omega^n$—in which case the wanted integral manifolds are given by Cartan's existence theorem—or that $\mathcal{V}$ does not admit any integral manifold on which $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$.

Drach observed that any PDE system is equivalent to either a first or a second order PDE system in one dependent variable. In chapter **5** we take a preliminary look at a single second order PDE in one dependent variable, and in particular investigate the presence of *singular* vector fields—that is, vector fields in $\mathcal{V}$ commuting modulo $\mathcal{V}$ with a greater number of vector fields than the average one does. There turn out to be either exactly two singular subsystems of $\mathcal{V}$, or none at all. If there are such, the PDE is *hyperbolic* if they are different, and *parabolic* if they coincide.

These observations give rise to the notion of *Monge characteristic subsystems* of the vector field system $\mathcal{V}$: a subsystem $\mathcal{M}$ of $\mathcal{V}$ is Monge if

(i) $\mathcal{M}$ is singular,
(ii) $\mathcal{M} \cap \mathcal{I} \neq 0$ for any maximal involution $\mathcal{I}$ of $\mathcal{V}$, and

(iii) $\mathcal{M} \cap \mathcal{W}$ is complete for any maximal complete subsystem $\mathcal{W}$ of $\mathcal{V}$.

The integral manifolds of $\mathcal{M} \cap \mathcal{W}$ are called *Monge characteristics*.

A special case is the *Cauchy charateristic subsystem* $\mathcal{C}(\mathcal{V})$ *of* $\mathcal{V}$:

$$\mathcal{C}(\mathcal{V}) := \{X \in \mathcal{V} \mid [X, \mathcal{V}] \subseteq \mathcal{V}\}.$$

$\mathcal{C}(\mathcal{V})$ is complete, and is included in any maximal complete subsystem of $\mathcal{V}$.

In chapter **6** we consider the integration of vector field systems satisfying $\dim \mathcal{V}' = \dim \mathcal{V} + 1$, which includes first order PDE systems in one dependent variable as a special case. Such a vector field system is essentially equivalent to a single pfaffian equation $\omega = 0$, which is solved by putting it into a canonical form:

$$\omega = 0 \iff dz - \sum_{i=1}^{n} p_i \, dx^i = 0.$$

The reduction procedure accomplishing this is a good demonstration of how powerful Lie's ideas are.

By Drach's classification there then remains to consider second order PDE systems in one dependent and n independent variables; the remainder of the monograph is devoted to the cases $n = 2$ and 3. The main method is

look for Monge systems and their first integrals!

At the outset there is no consideration at all of groups—but they will turn up anyway. The first example is the Lie pseudogroup of contact transformations, which consists of all local diffeomorphisms of the jet bundle $J^1(\mathbb{R}^n_x, \mathbb{R}_z)$ preserving the pfaffian equation $dz - \sum_{i=1}^n p_i \, dx^i = 0$. A general Lie pseudogroup is a family of local diffeomorphisms constituting the general solution of some PDE system, and being closed under composition whenever this is defined.

Chapter **7** discusses higher order contact transformations and prolongations of local diffeomorphisms to jet bundles.

In chapter **8** the general solution of the defining PDE system is supposed to depend on a *finite* number of parameters only, in which case the Lie pseudogroup is called a *local Lie group*. Acting on $\mathbb{C}^n$ and having r parameters, its elements are given by local diffeomorphisms

$$(x^1, \ldots, x^n) \mapsto (f^1(x^1, \ldots, x^n; a^1, \ldots, a^r), \ldots, f^n(x^1, \ldots, x^n; a^1, \ldots, a^r)),$$

or expressed more simply, $x \mapsto f(x;a) \equiv f_a(x)$. The group property means that whenever $f_a \circ f_b$ is defined, there is a $c = \phi(a,b)$ such that $f_a \circ f_b = f_c$. In this way there is induced a group action on the parameter space,

$$(a,b) \mapsto \phi(a,b),$$

giving this a Lie group structure in the modern sense—i.e., the parameter space acts on itself as a group.

In the action $(a,b) \mapsto \phi(a,b)$ the as can be regarded as 'variables' and the bs as 'parameters', or just the other way around. Accordingly two parameter groups arise, corresponding to right and left multiplication in modern terminology. And to these there correspond two Lie algebras of vector fields. Usually the attention is restricted to one of them, but for our applications it is essential to consider both.

Lie's original version of Lie group theory depends heavily on differential equations. In order to be able to study Lie groups and Lie algebras over more general fields than $\mathbb{C}$ and $\mathbb{R}$, 20$^{\text{th}}$ century mathematicians have worked hard to find purely algebraic foundations of the Lie theory. Unfortunately this has had the effect of cutting off Lie theory from differential equations—and therefore one has to struggle a bit in order to rebuild the bridge between these two disciplines. The same goes for Lie pseudogroups: what we will need here is Cartan's original version, rather than later algebraizations.

The study of hyperbolic second order PDEs requires the classification of Lie groups of dimension ≤ 3—which is derived in chapter **9**.

Cartan's local existence theorem ultimately depends on the Cauchy–Kowalewski theorem. The latter requires analyticity, and should therefore be avoided if possible. For instance, the integration of vector field systems $\mathcal{V}$ with $\dim \mathcal{V}' - \dim \mathcal{V} = 0$ or 1 is reduced to solving ODE systems only. Lie and his followers were not satisfied with general ODE systems either, but wanted to go one step further and make a reduction to what Lie called 'ausführbare Operationen'. An example is Lie's study of complete vector field systems admitting a nontrivial symmetry group, in which case he achieves a reduction to so called *Lie equations*. Originally these were defined as ODE systems corresponding to vector fields $\sum_{k=1}^{n} a^k(t) X_k(x)$, with $X_1(x), \ldots, X_n(x)$ forming a Lie algebra of vector fields. Later on Lie and Vessiot found that such systems can be characterized as ODE systems having the property that the general solution may be expressed as a certain function of a number of particular solutions.—All this is explained in chapter **10**, which also contains Vessiot's generalization to 2-dimensional Lie vector field systems.

Thus prepared we begin the discussion of second order PDEs in one dependent and two independent variables in chapter **11**. One item is the characterization of those vector field systems that arise from second order PDEs, and another is a sketch of Darboux's method for finding solutions by means of first integrals of the Monge systems.

Goursat found a remarkable classification of hyperbolic PDEs of the form

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right),$$

and having the property that each of the two Monge systems admits two or three functionally independent first integrals. Unfortunately he used ad hoc methods giving no clue whatsoever as to the underlying reasons.

Chapter **12** gives an account of Vessiot's theory of hyperbolic PDEs in one dependent and two independent variables, for which each of the Monge systems admits at least two independent first integrals—thus including the Goursat equations. Making use of these first integrals, the corresponding vector field systems are brought to a *finite number of canonical forms*. Remarkably the classification thus obtained ultimately depends on the classification of 2- and 3-dimensional Lie groups!

The Goursat equations are further investigated in chapter **13**.

While Vessiot's study of hyperbolic PDEs consists in a straightforward reduction to canonical forms, Cartan uses his solution of the equivalence problem in order to classify parabolic PDEs.

The equivalence problem is this: given two manifolds $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of the same dimension and having local structures $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ respectively (in our case vector field or pfaffian systems), is it possible to find local diffeomorphisms transforming the one structure into the other? Note that if

$$\phi_k : \boldsymbol{M}_1 \xrightarrow{\cong} \boldsymbol{M}_2, \qquad k = 1, 2,$$

are two such diffeomorphisms for which $\phi_2^{-1} \circ \phi_1$ and $\phi_1 \circ \phi_2^{-1}$ are defined, the latter are self-equivalences of $(\boldsymbol{M}_1, \boldsymbol{S}_1)$ and $(\boldsymbol{M}_2, \boldsymbol{S}_2)$ respectively. The family of self-equivalences of $(\boldsymbol{M}_k, \boldsymbol{S}_k)$ forms the *symmetry group* of $(\boldsymbol{M}_k, \boldsymbol{S}_k)$, and it is such symmetry groups that are the original reason for the introduction of the concept of Lie pseudogroup. If $(\boldsymbol{M}_1, \boldsymbol{S}_1)$ and $(\boldsymbol{M}_2, \boldsymbol{S}_2)$ are locally equivalent, their symmetry groups are obviously isomorphic, but the converse need not be true—imagine for instance two different structures which are so complicated that both their symmetry groups reduce to the identity.

Cartan's idea for solving the equivalence problem consists of two steps: first determine all local diffeomorphisms making the symmetry groups isomorphic, and then, among these, determine those which in addition are local equivalences. Now in most applications the first step alone suffices, and then we are reduced to studying equivalences of Lie pseudogroups.

Chapter **14** sketches Cartan's theory of Lie psedogroups, which hence is a preparation for the equivalence problem. The latter is dealt with in chapter **15**.

After this we are ready to tackle Cartan's 'five variable paper' in chapters **16** and **17**. First it is shown that parabolic PDEs for which the (double) Monge system admits at least two independent first integrals are equivalent to systems of two second order PDEs admitting a Cauchy characteristic vector field. These systems are 2-codimensional submanifolds of the 8-dimensional jet bundle $J^2(\mathbb{C}^2, \mathbb{C})$, and so are of dimension 6. The existence of one Cauchy characteristic vector field makes it possible to reduce the integration problem to one dimension less—hence the five variables.

Chapter **18** summarizes Cartan's work on second order PDE systems in one dependent and three independent variables. The idea is to simplify the structure equations of the corresponding pfaffian equations as far as possible, and then regard systems with the same reduced structure as *structurally equivalent*—while renouncing the detailed study of local equivalence. The reduced structure equations reveal for instance that all PDE systems consisting of at least two PDEs do admit singular subsystems, and the latter are then used in order to solve the integration problem by means of the *method of Monge*. The most surprising result is that *all* systems of *two* PDEs can be solved by a reduction to ODE systems!

According to the classification of Drach there then remains to look at second order PDE systems in one dependent and more than three independent variables—but that is quite a different story.

Convention: Only being interested in local properties, we let $\mathbb{R}^n_x$ denote a generic open neighbourhood of the origin in the n-dimensional Euclidean space, with $x^1, \ldots, x^n$ as local coordinates. This neighbourhood is not fixed, but may be shrunk whenever convenient.

2

PDE systems, pfaffian systems and vector field systems

The Norwegian mathematician Sophus Lie (1842–1899) is nowadays most known for his work on Lie groups and Lie algebras. But he regarded the latter—at least initially—mainly as a tool for understanding the theory of differential equations, which for him was the most important branch of mathematics. Or as he expressed it in his review paper 'Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung', *Leipziger Berichte*, 1895:

> *In der ganzen modernen Mathematik ist die Theorie der Differentialgleichungen die wichtigste Disziplin.*

Before entering into any serious work on differential equations, these must be formulated intrinsically. Previous to Lie this had been done in two special cases: first order ODE systems, and first order PDE systems in one dependent variable. So let us begin by looking at these, and then consider the very simplest type of PDE systems imaginable, namely those of Frobenius type.

2.1 ODE systems, vector fields and 1-parameter groups

Let $t, x^1, \ldots, x^n$ be coordinates for $\mathbb{R}^{n+1}_{t,x} = \mathbb{R}_t \times \mathbb{R}^n_x$, and let

$$\frac{dx_i}{dt} = f^i(t; x^1, \ldots, x^n), \qquad i = 1, \ldots, n, \tag{$\mathcal{S}$}$$

be a first order ODE system, with the f^i being smooth (i.e., C^∞) functions. Solving this is equivalent to determining the integral curves on which $dt \neq 0$ of the pfaffian system

$$\theta^i := dx^i - f^i(t; x)\, dt = 0, \qquad i = 1, \ldots, n. \tag{$\mathcal{P}$}$$

For let C be an integral curve of $\mathcal{P}$ on which $dt \neq 0$. Then the restrictions of the x^i to C can be expressed as functions of t, $x^i = x^i(t)$, and these satisfy

$$0 = dx^i - f^i(t;x)\,dt = \left(\frac{dx^i}{dt} - f^i(t;x)\right) dt \Longleftrightarrow \frac{dx^i}{dt} = f^i(t;x).$$

Conversely any solution of $\mathcal{S}$ clearly defines an integral curve of $\mathcal{P}$ on which $dt \neq 0$.

Notation. If $\theta^1, \ldots, \theta^n$ are one-forms on a certain manifold, $\mathcal{P} = (\theta^1, \ldots, \theta^n)$ denotes the module generated by the θ^i over the ring of smooth functions on the manifold. By abuse of notation $\mathcal{P}$ also denotes the corresponding system $\theta^i = 0, \quad i = 1, \ldots, n$. When we are thinking of $\mathcal{P}$ in the latter form, it is called a *pfaffian system.*

By definition the vector field

$$X = a^0(t;x)\frac{\partial}{\partial t} + a^1(t;x)\frac{\partial}{\partial x^1} + \cdots + a^n(t;x)\frac{\partial}{\partial x^n}$$

is dual to $\mathcal{P} = (\theta^1, \ldots, \theta^n)$ if and only if $\theta^i(X) = 0$ for $i = 1, \ldots, n$—i.e.,

$$0 = (dx^i - f^i(t;x)\,dt)\left(a^0\frac{\partial}{\partial t} + a^1\frac{\partial}{\partial x^1} + \cdots + a^n\frac{\partial}{\partial x^n}\right) = a^i - a^0 \cdot f^i(t;x)$$

$$\Longleftrightarrow a^i = a^0 \cdot f^i(t;x) \qquad \text{for } i = 1, \ldots, n.$$

Hence the dual of the pfaffian system $\mathcal{P} = (\theta^1, \ldots, \theta^n)$ is the 1-dimensional vector field system $\mathcal{V} = (X_0)$ generated by

$$X_0 = \frac{\partial}{\partial t} + \sum_{k=1}^{n} f^k(t;x)\frac{\partial}{\partial x^k}.$$

Notation. If $X_1, \ldots, X_q$ are vector fields on a manifold, the module of vector fields generated by the X_i over the ring of smooth functions on the manifold is denoted by $\mathcal{V} = (X_1, \ldots, X_q)$. Such a module is called a *vector field system.*

Integrating $\mathcal{V} = (X_0)$ is the same as finding the integral curves of X_0, i.e., those curves C for which the tangent space of C at an arbitrary point $c \in C$ is spanned by the value of X_0 at the point c. Since the coefficient of $\partial/\partial t$ in X_0 equals 1, t may be used as a local coordinate on such a curve C, which accordingly can be written as $t \mapsto (t; x^1(t), \ldots, x^n(t))$. Then the tangent vectors of C have the form

$$\frac{\partial}{\partial t} + \sum_{k=1}^{n} \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$$

for various t. Comparing this with X_0, we find that

$$\frac{dx^k}{dt} = f^k(t;x) \quad \text{for } k = 1, \dots, n.$$

Hence the integral curves of X_0 are in one-to-one correspondance with the solutions of $\mathcal{S}$.

Conclusion. *The following are equivalent:*

- *solving* $\mathcal{S}: dx^i/dt = f^i(t;x)$,
- *finding integral curves of* $\mathcal{P} = \left(dx^1 - f^1\, dt, \dots, dx^n - f^n\, dt\right)$ *on which* $dt \neq 0$,
- *integrating the vector field* $X_0 = \partial/\partial t + \sum_{k=1}^n f^k(t;x)\,\left(\partial/\partial x^k\right)$.

Let us consider the vector field aspect. From a geometric point of view it is natural to try to rectify X_0—that is, to find a coordinate transformation

$$\Phi : \mathbb{R}_s \times \mathbb{R}^n_y \xrightarrow{\cong} \mathbb{R}_t \times \mathbb{R}^n_x$$

making X_0 equal to $\Phi_*(\partial/\partial s)$. Then the integral curves are given by

$$y^i = \text{ constant} \qquad \text{for } i = 1, \dots, n.$$

Suppose that the wanted Φ is of the form

$$\Phi : \begin{cases} t = s, \\ x^i = \xi^i(s; y^1, \dots, y^n), \qquad i = 1, \dots, n, \end{cases}$$

with the inverse

$$\Phi^{-1} : \begin{cases} s = t, \\ y^i = \eta^i(t; x^1, \dots, x^n), \qquad i = 1, \dots, n. \end{cases}$$

Then

$$\Phi_*\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t} + \sum_{i=1}^n \frac{\partial \xi^i}{\partial s}(t; \eta^1(t;x), \dots, \eta^n(t;x))\, \frac{\partial}{\partial x^i},$$

so that $\Phi_*(\partial/\partial s) = X_0$ if and only if

$$\frac{\partial \xi^i}{\partial s}(t; \eta^1(t;x), \dots, \eta^n(t;x)) = f^i(t; x^1, \dots, x^n) \qquad \text{for } i = 1, \dots, n,$$

or equivalently

$$\frac{\partial \xi^i}{\partial s}(s; y^1, \dots, y^n) = f^i(s; \xi^1(s;y), \dots, \xi^n(s;y)) \qquad \text{for } i = 1, \dots, n.$$

In order to ensure that Φ becomes a local diffeomorphism we impose the initial conditions

$$\xi^i(0; y^1, \dots, y^n) = y^i \quad \text{for } i = 1, \dots, n,$$

because then

$$\frac{\partial(t; x^1, \dots, x^n)}{\partial(s; y^1, \dots, y^n)} = 1 \quad \text{when } t = s = 0.$$

So Φ is determined by the system

$$\frac{\partial \xi^i}{\partial s}(s; y^1, \dots, y^n) = f^i(s; \xi^1(s; y), \dots, \xi^n(s; y)),$$
$$\xi^i(0; y^1, \dots, y^n) = y^i, \qquad i = 1, \dots, n.$$

If we regard s as an honest variable and the y^i as parameters, this is a parametrized ODE system. The fundamental existence and uniqueness theorem for such systems (due to Euler, Cauchy, Lipschitz, Picard, ...) then shows that there is a unique local solution Φ.

Let us state this in a slightly more general form as the **local rectification lemma for vector fields.**

Lemma 2.1.1. *Let $X = \sum_{i=0}^n a^i(x)\,(\partial/\partial x^i)$ be a smooth vector field defined near $x = 0$, with $a^0(0) \neq 0$. Then there is a unique local diffeomorphism*

$$\Phi : \mathbb{R}^{n+1}_y \xrightarrow{\cong} \mathbb{R}^{n+1}_x$$

such that

$$\Phi_*(\partial/\partial y^0) = X \quad \text{and} \quad x^i(0, y^1, \dots, y^n) = y^i \quad \text{for } i = 1, \dots, n.$$

□

Notation. A *first integral* of a vector field X is a nonconstant function f satisfying $Xf = 0$.

Using this notion the rectification lemma says that the vector field $X = \sum_{i=0}^n a^i(x)\,(\partial/\partial x^i)$ has a uniquely determined set $\{y^1, \dots, y^n\}$ of local first integrals, satisfying

$$y^i(0, x^1, \dots, x^n) = x^i \quad \text{for } i = 1, \dots, n.$$

Return to the vector field $X_0 = \partial/\partial t + \sum_{k=1}^n f^k(t; x)\,(\partial/\partial x^k)$. In the new coordinates $s, y^1, \dots, y^n$,

$$X_0 = \frac{\partial}{\partial s},$$