Elliptic Functions

In its first six chapters this text seeks to present the basic ideas and properties of the Jacobi elliptic functions as an historical essay, an attempt to answer the fascinating question: 'what would the treatment of elliptic functions have been like if Abel had developed the ideas, rather than Jacobi?' Accordingly, it is based on the idea of inverting integrals which arise in the theory of differential equations and, in particular, the differential equation that describes the motion of a simple pendulum.

The later chapters present a more conventional approach to the Weierstrass functions and to elliptic integrals, and then the reader is introduced to the richly varied applications of the elliptic and related functions. Applications spanning arithmetic (solution of the general quintic, the representation of an integer as a sum of three squares, the functional equation of the Riemann zeta function), dynamics (orbits, Euler's equations, Green's functions), and also probability and statistics, are discussed.

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and the late

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W. F. Eberlein's original was dedicated to Patrick, Kathryn, Michael, Sarah, Robert, Mary and Kristen;

I should like to add:

To Sarah, Mark and Nicholas

(J. V. Armitage).

For some minutes Alice stood without speaking, looking out in all directions over the country – and a most curious country it was. There were a number of tiny little brooks, running straight across it from side to side, and the ground between was devided up into squares by a number of little green hedges, that reached from brook to brook. 'I declare it's marked out just like a large chess-board!' Alice said at last.

Lewis Carroll, *Through the looking Glass*

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Preface

This is essentially a prolegomenon to the Partial Preface and serves primarily to place in a proper context the contents of this book and how they relate to the original six chapters, to which it refers.

Those six chapters, originally by W. F. Eberlein, sought to relate the ideas of Abel to the later work of Jacobi and concluded with the transformation theory of the theta functions. The first chapter began with the differential equation associated with the motion of a simple pendulum, very much in the tradition of Greenhill's *'Applications of Elliptic Functions'* (1892), but much influenced by the spirit of modern analysis. (Greenhill's obituary reads that 'his walls (were) festooned with every variety of pendulum, simple or compound.'[1]) The version given here is inspired by those early chapters and, apart from the addition of illustrative examples and extra exercises, is essentially unchanged.

The present account offers six additional chapters, namely 7 to 12, together with an Appendix, which seek to preserve the essentials and the spirit of the original six, insofar as that is possible, and which include an account of the Weierstrass functions and of the theory of elliptic integrals in Chapters 7 and 8. There follows an account of applications in (mainly classical) geometry (Chapter 9); in algebra and arithmetic – the solution of the quintic in Chapter 10, and sums of three squares, with references to the theory of partitions and other arithmetical applications in Chapter 11); and finally, in classical dynamics and physics, in numerical analysis and statistics and another arithmetic application (Chapter 12). Those chapters (9 to 12), were inspired by Projects offered by Fourth Year M. Math. undergraduates at Durham University, who chose topics on which to work, and they are offered here partly to encourage similar work. Finally the Appendix includes topics from its original version and then an application to the Riemann zeta function. There are extensive references to further reading in topics outside the scope of the present treatment.

[1] J. R. Snape supplied this quotation.

Original partial preface

(Based on W. F. Eberlein's preface to chapters 1 to 6, with some variations and additions)

Our thesis is that on the untimely death of Abel in 1829, at the age of 26, the theory of elliptic functions took a wrong turning, or at any rate failed to follow a very promising path. The field was left (by default) to Jacobi, whose 18^{th} century methods lacked solid foundations until he finally reached the firm ground of theta functions. Thence emerged the notion of pulling theta functions out of the air and then defining the Jacobian elliptic functions as quotients of them.

All that is, of course, rigorous, but perhaps it puts the theta function cart before the elliptic functions horse! For example, the celebrated theta function identity

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^8 + 16q\prod_{n=1}^{\infty}(1+q^{2n})^8 = \prod_{n=1}^{\infty}(1+q^{2n-1})^8$$

looks impressive and was described by Jacobi as 'aequatio identica satis abstrusa', but if one starts in the historical order with elliptic functions, it reduces to the relatively trivial, if perhaps more inscrutable, identity $k^2 + (1-k^2) = 1$. (See Chapter 4.)

In this book we shall apply *Abel's methods*, supplemented by the rudiments of complex variable theory, to *Jacobi's functions* to place the latter's elegance upon a natural and rigorous foundation.

A pedagogical note may be helpful at this point; we prefer to motivate theorems and proofs. Influenced by the writings of George Polya, we have tried to motivate theorems and proofs by 'induction and analogy' and by 'plausible inference' on all possible occasions. For example, the addition formulae for the Jacobian elliptic functions are usually pulled out of a hat, but (cf the book by Bowman [12]) in Chapter 2 we *guess* the basic addition formula for $cn(u+v,k)$, $0<k<1$, by interpolating between the known limiting cases $k=0$ (when $cn(u+v,0)=\cos(u+v)$) and $k=1$ (when $cn(u+v,1)=\sec h(u+v)$). We have tried to adopt similar patterns of plausible reasoning throughout.

Acknowledgements

With grateful thanks to Sarah, who typed the preliminary version, and to eight fourth year undergraduates at Durham University, whose projects in the applications of elliptic functions I supervised and from whom I learnt more than they did from me.

Acknowledgements are made throughout the book, as appropriate, to sources followed, especially in Chapters 8 to 12, which are devoted to the applications of elliptic functions. The work of the students referred to in the preceding paragraphs was concerned with projects based on such applications and used the sources quoted in the text. For example the work on the solution of the general quintic equation (Chapter 10) relied on the sources quoted and on individual supervisions in which we worked through and interpreted the references cited, with additional comments as appropriate. I should like to acknowledge my indebtedness to those texts and to the students who worked through them wih me. I should also like to acknowledge my indebtedness to Dr Cherry Kearton, with whom I supervised a project on cryptography and elliptic curves.

I lectured on elliptic functions (albeit through a more conventional approach and with different emphases and without most of the applications offered here) since the nineteen-sixties at King's College London and at Durham University. Again, I should like to express my gratitude to students who attended those lectures, from whose interest and perceptive questions I learnt a great deal. Over the years I built up an extensive collection of exercises, based on the standard texts (to which reference is made in what follows) and on questions I set in University examinations. Those earlier courses did not involve the applications offered here, nor did they reflect the originality of the unconventional approach offered here, as in the early Chapters by W. F. Eberlein, but I should like nevertheless to acknowledge my indebtedness to collections of exercises due to my colleagues at Durham, Professor A. J. Scholl and Dr J. R. Parker, though

I should add that neither of them is to be blamed for any shortcomings to be found in what follows.

I would like to express my thanks to Caroline Series of the London Mathematical Society, who invited me to complete Eberlein's six draft chapters for the Student Texts Series. I would also like to thank Roger Astley of Cambridge University Press for his patient encouragement over several years, and Carol Miller, Jo Bottrill and Frances Nex for their most helpful editorial advice.