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J. V. Armitage and W. F. Eberlein
Excerpt
More information

1
The 'simple' pendulum

1.1 The pit and the pendulum

The inspiration for this introduction is to be found in Edgar Allen Poe's *The Pit and the Pendulum, The Gift* (1843), reprinted in *Tales of Mystery and Imagination*, London and Glasgow, Collins.

A simple pendulum consists of a heavy particle (or 'bob') of mass m attached to one end of a light (that is, to be regarded as weightless) rod of length l, (a constant). The other end of the rod is attached to a fixed point, O. We consider only those motions of the pendulum in which the rod remains in a definite, vertical plane. In Figure 1.1, P is the particle drawn aside from the equilibrium position, A. The problem is to determine the angle, θ, measured in the positive sense, as a function of the time, t.
The length of the arc AP is $l\theta$ and so the velocity of the particle is

$$\frac{\mathrm{d}}{\mathrm{d}t}(l\theta) = l\frac{\mathrm{d}}{\mathrm{d}t}\theta,$$

and it is acted upon by a downward force, mg, whose tangential component is $-mg\sin\theta$. Newton's Second Law then reads

$$m\left[\frac{\mathrm{d}}{\mathrm{d}t}l\frac{\mathrm{d}\theta}{\mathrm{d}t}\right] = -mg\sin\theta,$$

that is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0. \tag{1.1}$$

If one sets $x = (g/l)^{1/2}\, t$, (so that x and t both stand for 'time', but measured in different units), then (1.1) becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \sin\theta = 0. \tag{1.2}$$

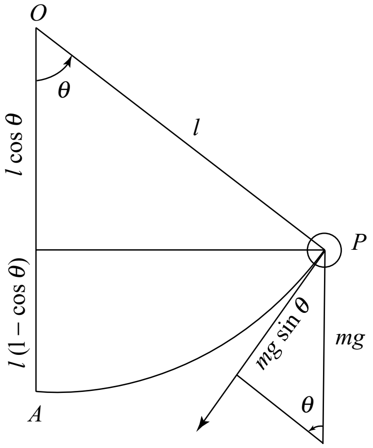

Figure 1.1 The simple pendulum.

Equation (1.2) has a long history (of some 300 years at least), but a rigorous treatment of it is by no means as straightforward as one might suppose (whence the 'pit'!). As one example of the pitfalls we might encounter and must try to avoid in our later development of the subject, let us begin with the familiar linearization of (1.2) obtained by supposing that θ is small enough to permit the replacement of $\sin\theta$ by θ. We obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \theta = 0 \tag{1.3}$$

and the general solution of (1.3) is

$$\theta = A\cos x + B\sin x,$$

where A, B are constants, and the motion is periodic with x having period 2π and t having period $2\pi(l/g)^{1/2}$. The unique solution of Equation (1.3) satisfying the initial conditions $\theta(0) = 0$, $\theta'(0) = 1$, where θ' denotes $\frac{\mathrm{d}\theta}{\mathrm{d}x}$, is $\theta(x) = \sin x$.

Equation (1.3) is the familiar equation defining the simple harmonic motion of a unit mass attached to a spring with restoring force $-\theta$, where θ is the displacement from the equilibrium position at the time x. As such it is discussed in elementary calculus and mechanics courses; so what can go wrong?

Since the independent variable, x, is absent from (1.3) the familiar procedure is to put $v = \theta'$ and then

$$\theta'' = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}\theta};$$

so that (1.3) becomes $v\dfrac{\mathrm{d}v}{\mathrm{d}\theta} + \theta = 0$, or

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\frac{1}{2}v^2 + \frac{1}{2}\theta^2\right\} = 0;$$

that is, $\frac{1}{2}v^2 + \frac{1}{2}\theta^2 \equiv C$, where C is a constant.

Now $\frac{1}{2}v^2$ is the kinetic energy and $\frac{1}{2}\theta^2$ is the potential energy of the system and so we recover the familiar result that the total energy is constant (the sum is the energy integral). On inserting the initial conditions, namely $\theta = 0$ and $v = 1$ when $x = 0$, we obtain $C = 1/2$ and so

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 = 1 - \theta^2. \tag{1.4}$$

Clearly the solution already found for (1.3), $\theta(x) = \sin x$, satisfies (1.4), but, in passing from (1.3) to (1.4), we have picked up extra solutions. For example, if we write

$$\theta(x) = \begin{cases} \sin x, & x < \dfrac{\pi}{2}, \\ 1, & x \geq \dfrac{\pi}{2}, \end{cases} \tag{1.5}$$

then (1.5) is a C^1 solution[1] of (1.4) for all x, but not of (1.3), when $x > \frac{\pi}{2}$. Physically, the solution (1.5) corresponds to the linearized pendulum 'sticking' when it reaches maximum displacement at time $x = \pi/2$. (Of course we chose θ to be small and so the remark is not applicable to the pendulum problem, but it is relevant in what follows.)

We shall have to be aware of the pitfalls presented by that phenomenon, when we introduce the elliptic functions in terms of solutions of differential equations. It falls under the heading of 'singular solutions'; for a clear account see, for example, Agnew (1960), pp. 114–117.

Exercise 1.1

1.1.1 Re-write (1.4) in the form $v^2 = 1 - \theta^2$ and observe that $\frac{\mathrm{d}v}{\mathrm{d}\theta}$ is infinite at $\theta = \pm 1$.

[1] Here and in what follows we use the notation $f(x) \in C^k(I)$ to mean that f is a complex valued function of the real variable x defined on the open interval $I : a < x < b$ and having k continuous derivatives in I (with obvious variations for closed or half-open intervals). If $k = 0$, the function is continuous.

1.2 Existence and uniqueness of solutions

So far, we have not proved that Equation (1.2) has any solutions at all, physically obvious though that may be. We now address that question.

Set $\omega = \frac{\mathrm{d}\theta}{\mathrm{d}x}$ and re-write (1.2) as a first order autonomous[2] system

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}x} &= \omega \equiv f(\theta, \omega),\\ \frac{\mathrm{d}\omega}{\mathrm{d}x} &= -\sin\theta \equiv g(\theta, \omega).\end{aligned} \tag{1.6}$$

The functions $f(\theta, \omega)$, $g(\theta, \omega)$ are both in $C^\infty(\mathbb{R}^2)$ and the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial \theta} & \dfrac{\partial g}{\partial \theta} \\ \dfrac{\partial f}{\partial \omega} & \dfrac{\partial g}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 0 & -\cos\theta \\ 1 & 0 \end{pmatrix}$$

is bounded on $\mathbb{R}^2$.

We can now appeal to the theory of ordinary differential equations (see Coddington & Levinson (1955), pp. 15–32, or Coddington (1961)) to obtain the following.

Theorem 1.1 *The system (1.6) corresponding to the differential equation (1.2) has a solution $\theta = \theta(x)$, $(-\infty < x < \infty)$, such that $\theta(a) = A$ and $\theta'(a) = B$, where a, A and B are arbitrary real numbers. Moreover, that solution is unique on any interval containing a.*

Note that Equation (1.2) implies that the solution $\theta(x)$, whose existence is asserted in Theorem 1.1, is in $C^\infty(\mathbb{R})$.

The result of Theorem 1.1 is fundamental in what follows.

1.3 The energy integral

On multiplying Equation (1.1) by $ml^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$, we obtain

$$ml^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0,$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\left(l\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + mgl(1-\cos\theta)\right] = 0.$$

[2] Equations (1.6) are the familiar way of writing a second order differential equation as a system of linear differential equations; the system is said to be autonomous when the functions $f(\theta, \omega)$, $g(\theta, \omega)$ do not depend explicitly on x (the time).

Hence,

$$\frac{1}{2}m\left(l\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 + mgl(1-\cos\theta) = E, \tag{1.7}$$

where E is a constant. By referring to Figure 1.1, we see that the first term in (1.7) is the kinetic energy of the pendulum bob and the second term is the potential energy, measured from the equilibrium position, A. The energy required to raise the pendulum bob from the lowest position ($\theta = 0$) to the highest possible position ($\theta = \pi$) (though not necessarily attainable – that depends on the velocity at the lowest point) is $2mgl$. So we may write

$$E = k^2(2mgl), \quad k \geq 0. \tag{1.8}$$

Clearly, we can obtain any given $k \geq 0$ by an appropriate choice of the initial conditions. We now assume $0 < k < 1$ (oscillatory motion).

It will be helpful later to look at (1.8), on the assumption $0 < k < 1$, from a slightly different point of view, as follows (see Exercise 1.3.2)

Suppose that $v = v_0$ and $\theta = \theta_0$, when $t = 0$. Then we have

$$\frac{1}{2}\left(v_0^2 - v^2\right) = gl(1-\cos\theta);$$

that is

$$v^2 = v_0^2 - 4gl\sin^2\frac{\theta}{2}.$$

On writing $v = l\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$, $h^2 = \dfrac{g}{l}$, we obtain

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = 4h^2\left(\frac{v_0^2}{4gl} - \sin^2\frac{\theta}{2}\right). \tag{1.9}$$

On comparing (1.7), (1.8) and (1.9), we see that $k^2 = v_0^2/(4gl)$.

Recall our assumption that $0 < k < 1$, that is that $v_0^2 < 4gl$; so that the bob never reaches the point given by $\theta = \pi$ and the motion is, accordingly, oscillatory (which is what one would expect of a pendulum). Comparison with (1.9) suggests that we write $k = v_0/(2\sqrt{(gl)}) = \sin\alpha/(2)$, where $0 < \alpha < \pi$, and re-introduce the normalized time variable $x = (g/l)^{1/2}\ t$, to obtain (cf. (1.9)).

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 = 4\left(k^2 - \sin^2\frac{\theta}{2}\right) = 4\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right). \tag{1.10}$$

It is clear that any solution of (1.2) corresponding to $k, 0 < k < 1$, is a solution of (1.10), but is the converse true? (In other words have we produced a situation similar to that given by (1.5)?)

One sees immediately that $\theta \equiv \pm\alpha + 2\pi n$ is a solution of (1.10), but not of (1.2). Are there other solutions of (1.10) analogous to the solution (1.5) of (1.3)?

Recall that $\theta''\left(\pi/2\right)$ does not exist in (1.5); so consider the $C^2(-\infty, \infty)$ solutions of (1.10). If we reverse the argument that led to (1.10), we obtain

$$(\theta'' + \sin\theta) \cdot \theta' = 0,$$

and so

$$\theta'' + \sin\theta = 0,$$

except possibly on the set Γ defined by

$$\Gamma = \{x \in \mathbb{R} | \theta'(x) = 0\}.$$

Now (1.10) implies $\theta'(x) = 0$ only when

$$\theta(x) = \pm\alpha + 2\pi n, \quad n \in \mathbb{Z}.$$

So there are two cases to consider:

Case 1: Γ contains interior points;
Case 2: Γ contains no interior points.

Theorem 1.2 *In Case 1,* $\Gamma = \mathbb{R}$ *and then* $\theta = \pm\alpha + 2\pi n$*. In Case 2,* $\theta'' + \sin\theta = 0$ *for all x.*

Proof Suppose that Case 1 holds and assume that $\Gamma \neq \mathbb{R}$. We argue indirectly.

By hypothesis, there exists an interval $[a, b]$, $(-\infty < a < b < +\infty)$, on which $\theta' = 0$ and a point, c, such that $\theta'(c) \neq 0$. Then either $c < a$ or $c > b$. Suppose that $c > b$ and write $d = \sup\{x \in \mathbb{R} | b \leq x, \theta'(s) = 0,\ a \leq s \leq x\}$. Then d $\leq c$ and $\theta'(d) = 0$ by the continuity of θ' (recall that we are considering $C^2(-\infty, +\infty)$ solutions). It follows that $d < c$ and $\theta'(x) = 0$ for $a \leq x \leq d$.

Moreover, given $\varepsilon > 0$, there exists $x \in [d, d + \varepsilon]$, with $\theta'(x) \neq 0$. It follows that there exists a sequence $\{x_n\}$, with $x_n + d \neq 0$ and $\theta'(x_n) \neq 0$ for every n. But then $\theta''(x_n) = -\sin\theta(x_n)$, whence, by the continuity of θ'',

$$\theta''(d) = \lim_{n\to\infty} \theta''(x_n) = -\lim_{n\to\infty} \sin\theta(x_n) = -\sin\theta(d) = -\sin(\pm\alpha) \neq 0.$$

But $\theta'(x) = 0$ for $a \leq x \leq d$ implies $\theta''(x) = 0$ for $a < x < d$, whence $\theta''(d) = 0$, by the continuity of θ''. So we have obtained the contradiction we sought.

The case $c < a$ is similar and can be reduced to the case $c > b$ by making the substitution $x \to -x$, under which both (1.2) and (1.10) are invariant.

The remaining statements when Case 1 holds are trivial.

Finally, in Case 2, we must show that if $a \in \Gamma$, then $\theta''(a) + \sin\theta(a) = 0$. Now since a is a boundary point of Γ, there exists a sequence $\{x_n\} \subset R - \Gamma$ such that $x_n \to a$. Since θ, θ'' are continuous, $\theta''(a) + \sin\theta(a) = \lim_{n\to\infty}\{\theta''(x_n) + \sin\theta(x_n)\} = 0$.

It follows that the only C^2 solutions of (1.10) which are not solutions of (1.2) are the singular solutions $\theta = \pm\alpha + 2\pi n$. Later, we shall construct a 'sticking solution' of (1.10), analogous to (1.5).

That completes the proof of Theorem 1.2.

We conclude this section with a result that is physically obvious.

Proposition 1.1 *Let θ be a solution of (1.10) such that $-\pi \le \theta(a) \le \pi$, for some a. Then $-\alpha \le \theta(x) \le \alpha$, for $-\infty < x < +\infty$.*

Proof With respect to the variable x, the energy equation (1.7) reads

$$\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 + (1 - \cos\theta) = \frac{E}{mgl} = 2k^2 = 2\sin^2\frac{\alpha}{2}. \tag{1.11}$$

Let $\theta(a) = A$. Then $2\sin^2(A/2) = 1 - \cos A \le 2\sin^2(\alpha/2)$ and $-\pi \le A \le \pi$ together imply $-\alpha \le A \le \alpha$.

Suppose that $\theta(b) > \alpha$, for some b. Then, by the Intermediate Value Theorem, there exists c such that $a < c < b$ and $\alpha < \theta(c) < \pi$. But then (1.10) implies $\theta'(c)^2 < 0$ – a contradiction. Hence $\theta(x) \le \alpha$ for all x. The proof that $\theta(x) \ge -\alpha$ for all x is similar and is left as an exercise.

The essential content of Proposition 1.1 is that, without loss of generality, we may and shall assume henceforth that all solutions θ of (1.10) satisfy $-\alpha \le \theta(x) \le \alpha$, for all x, since θ and $\theta + 2\pi n$ are simultaneously solutions of (1.2) and (1.10). Note that $0 < k \ll 1$ ('k is very much less than 1') implies $\alpha \ll 1$, whence $|\theta| \le \alpha \ll 1$ and then (1.3) is a good approximation to (1.2).

Exercises 1.3

1.3.1 Show that the changes of variable

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}, \; \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

applied to Equation (1.1) yield the energy integral (1.7).

1.3.2 Starting from the equation of energy for the simple pendulum, namely

$$\dot{\theta}^2 = -4g \sin^2 \frac{\theta}{2} + \text{constant},$$

suppose that when the pendulum bob is at its lowest point, the velocity v_0 satisfies

$$\frac{v_0^2}{2g} = \frac{l^2 \dot{\theta}^2}{2g} = h.$$

Show that the energy equation is

$$l^2 \dot{\theta}^2 = 2gh - 4gl \sin^2 \frac{\theta}{2}$$

and then write $y = \sin(\theta/2)$ to obtain the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{g}{l}(1 - y^2)\left(\frac{h}{2l} - y^2\right).$$

Suppose that the motion of the pendulum is oscillatory, that is $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ for some $y < 1$, whence $0 < h/2l < 1$. Write $h = 2lk^2$ and so obtain

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{gk^2}{l}\left(1 - k^2 \frac{y^2}{k^2}\right)\left(1 - \frac{y^2}{k^2}\right). \tag{1.12}$$

Replace y/k by y to obtain the Jacobi normal form (1.14), in Section 1.4, below, where the significance of this exercise will become apparent.

1.3.3 Suppose that the motion is of the circulatory type in which $h > 2l$ (so that the bob makes complete revolutions). If $2l = hk^2$ (so that the k for the oscillatory motion is replaced by $1/k$), and then, again $0 < k < 1$. Show that Equation (1.12) now reads

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{g}{lk^2}(1 - y^2)(1 - k^2 y^2).$$

1.4 The Euler and Jacobi normal equations

We have already exhibited Equation (1.10) in two different forms, and in this section we review all that and make some classical changes of variable (due originally to Euler and Jacobi) in the light of our earlier preview.

First we write (following Euler)

$$\phi = \arcsin\left(k^{-1} \sin \frac{\theta}{2}\right). \tag{1.13}$$

The map $\theta \mapsto \phi$ is a homeomorphism[3] of $[-\alpha, \alpha]$ onto $[-\pi/2, \pi/2]$ and a C^∞ – diffeomorphism of $(-\alpha, \alpha)$ onto $(-\pi/2, \pi/2)$. (Note that the requirement in the latter case, that the interval $(-\alpha, \alpha)$ be an *open* interval, is essential, since $\frac{d\phi}{d\theta}$ is meaningless when $\theta = \pm\alpha$ (and so $\phi = \pm\pi/2$)).

Now differentiate $\sin(\theta/2) = k \sin\phi$ with respect to x to obtain $\frac{1}{2}\cos\frac{\theta}{2} \cdot \frac{d\theta}{dx} = k\cos\phi \cdot \frac{d\phi}{dx}$, whence

$$\left(\frac{d\theta}{dx}\right)^2 = \frac{4k^2\cos^2\phi}{\cos^2\dfrac{\theta}{2}}\left(\frac{d\phi}{dx}\right)^2 = \frac{4k^2(1-\sin^2\phi)}{1-\sin^2\dfrac{\theta}{2}}\left(\frac{d\phi}{dx}\right)^2.$$

Hence (1.10) becomes

$$\frac{4k^2(1-\sin^2\phi)}{1-k^2\sin^2\phi}\left(\frac{d\phi}{dx}\right)^2 = 4k^2(1-\sin^2\phi),$$

that is

$$\left(\frac{d\phi}{dx}\right)^2 = 1 - k^2\sin^2\phi, \quad \left(-\frac{\pi}{2} < \phi < \frac{\pi}{2}\right). \tag{1.14}$$

Equation (1.14) is Euler's normal form; we shall see later that it remains valid when $\phi = \pm\pi/2$.

To obtain Jacobi's normal form, we can use the substitution (due to Jacobi)

$$y = \sin\phi = k^{-1}\sin\frac{\theta}{2}. \tag{1.15}$$

Then the increasing function $\theta \mapsto y$ is a C^∞ diffeomorphism of $[-\alpha, \alpha]$ onto $[-1, 1]$ and this time the end-points may be included. On differentiating (1.15) with respect to x, we obtain

$$\frac{dy}{dx} = \frac{k^{-1}}{2}\cos\frac{\theta}{2}\frac{d\theta}{dx}$$

and so

$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= \frac{k^{-2}}{4}\left(1-\sin^2\frac{\theta}{2}\right)4\left(k^2-\sin^2\frac{\theta}{2}\right)\\ &= \left(1-\sin^2\frac{\theta}{2}\right)\left(1-k^{-2}\sin^2\frac{\theta}{2}\right).\end{aligned}$$

[3] Recall that a *homeomorphism is a* one-to-one continuous map whose inverse exists throughout its range and a C^n*-diffeomorphism* is a bijective, n – times continuously differentiable map.

So we see that (1.10) becomes

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = (1-y^2)(1-k^2y^2), \quad -1 \leq y \leq 1, \tag{1.16}$$

which is *Jacobi's normal form*. (Compare with Exercise 1.3.2, and note that $k = (\sin\theta/2)y|_\alpha < 1$.)

1.5 The classical formal solutions of (1.14)

Denote by $\theta_0 = \theta_0(x|k)$ (the notation exhibits the dependence of θ_0 on k as well as on x) that solution of (1.2) such that $\theta_0(0) = 0$ and $\theta_0'(0) = 2k$, $0 < k < 1$. Then in the notation of (1.8), we have $E/(mgl) = \frac{1}{2}(2k)^2 + 0 = 2k^2$ and so θ_0 satisfies (1.10) with the same k. So at time $t = 0$ the bob is in its lowest position and is moving counter-clockwise with velocity sufficient to ensure that $\theta = \alpha$ when $t = T/4$, where T is the period of the pendulum (the time required for a complete swing). All that is plausible on physical grounds; but we must give a proof.

The Euler substitution

$$\phi = \arcsin\left(k^{-1}\sin\frac{\theta_0}{2}\right)$$

yields $\phi = 0$ and $\frac{\mathrm{d}\phi}{\mathrm{d}x} = 1$ when $x = 0$. We shall try to solve (1.10) under those initial conditions.

In some neighbourhood of $x = 0$, we must take the positive square root in (1.14) to obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \sqrt{1-k^2\sin^2\phi} \tag{1.17}$$

and we note that that is > 0, provided that x is sufficiently small. It follows that

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = \frac{1}{\sqrt{1-k^2\sin^2\phi}}, \tag{1.18}$$

provided that ϕ is sufficiently small. The solution of (1.18) under the given initial conditions is

$$x = \int_0^\phi \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{1.19}$$