

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

AUSTRALIAN MATHEMATICAL SOCIETY LECTURE SERIES

Editor-in-Chief: Professor J.H. Loxton, School of Mathematics, Physics, Computing and Electronics, Macquarie University, NSW 2109, Australia

Editors: Professor C.C. Heyde, School of Mathematical Sciences, Australian National University, Canberra, ACT 0200, Australia

Associate Professor W.D. Neumann, Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

Associate Professor C.E.M Pearce, Department of Applied Mathematics, University of Adelaide, SA 5005, Australia

- 3 Introduction to the Analysis of Metric Spaces, J. R. GILES
- 5 2-Knots and their Groups, J. HILLMAN
- 6 The Mathematics of Projectiles in Sport, N. DE MESTRE
- 7 The Petersen Graph, D. A. HOLTON & J. SHEEHAN
- 8 Low Rank Representations and Graphs for Sporadic Groups,
C. PRAEGER & L. SOICHER
- 9 Algebraic Groups and Lie Groups, G. LEHRER (ed)
- 10 Modelling with Differential and Difference Equations,
G. FULFORD, P. FORRESTER & A. JONES
- 11 Geometric Analysis and Lie Theory in Mathematics and Physics,
A. L. CAREY & M. K. MURRAY (eds)
- 12 Foundations of Convex Geometry, W. A. COPPEL
- 13 Introduction to the Analysis of Normed Linear Spaces, J. R. GILES
- 14 The Integral, LEE PENG YEE & R. VÝBORNÝ
- 15 Geometric Approaches to Differential Equations,
P. VASSILIOU, I. LISLE (eds)

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee

The National Institute of Education, Singapore

Rudolf Vyborný

The University of Queensland



Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock
Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS

The Edinburgh Building, Cambridge, CB2 2RU, UK

40 West 20th Street, New York, NY 10011-4211, USA

10 Stamford Road, Oakleigh, Melbourne 3166, Australia

Ruiz de Alarcón 13, 28014 Madrid, Spain

www.cup.cam.ac

www.cup.org

© Lee Peng Yee, Rudolf Vyborný 2000

**This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.**

First published 2000

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this book is available from the British Library

ISBN 0 521 77968 5 paperback

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

Contents

<i>Preface</i>	<i>page</i> viii
<i>List of Symbols</i>	xi
1 Introduction	1
1.1 Historical remarks	1
1.2 Notation and the Riemann definition	3
1.3 Basic theorems, upper and lower integrals	6
1.4 Differentiability, continuity and integrability	10
1.5 Limit and R-integration	16
1.6 Exercises	18
2 Basic Theory	22
2.1 Introduction	22
2.2 Motivation	22
2.3 Cousin's lemma	25
2.3.1 Applications of Cousin's lemma	26
2.4 The definition	29
2.5 Basic theorems	32
2.6 The Fundamental Theorem of calculus	46
2.7 Consequences of the Fundamental Theorem	50
2.8 Improper integrals	55
2.9 Integrals over unbounded intervals	59
2.10 Alternative approach to integration over unbounded intervals	64
2.11 Negligible sets	66
2.12 Complex valued function	69
2.13 Exercises	71
3 Development of the Theory	76
3.1 Equivalent forms of the definition	76
3.2 Henstock's lemma	81
3.3 Functions of bounded variation	83

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

vi	<i>Contents</i>	
3.4	Absolute integrability	86
3.5	Limit and KH-integration	88
3.6	Absolute continuity	100
3.7	Equiintegrability	104
3.7.1	The second mean value theorem	108
3.8	Differentiation of integrals	110
3.9	Characterization of the KH-integral	112
3.10	Lebesgue points, approximation by step functions	115
3.11	Measurable functions and sets	117
3.11.1	A non-measurable set	126
3.12	The McShane integral	127
3.12.1	A short proof	135
3.13	The Lebesgue integral	135
3.13.1	F. Riesz' definition	139
3.13.2	Quick proofs	140
3.14	Differentiation almost everywhere	140
3.15	Exercises	145
4	The SL-integral	151
4.1	The strong Luzin condition	151
4.2	SL-integration	155
4.3	Limit and SL-integration	162
4.4	Equivalence with the KH-integral	169
4.5	Exercises	171
5	Generalized AC Functions	175
5.1	Prologue	175
5.2	Uniformly AC functions	177
5.3	AC^* and VB^* on a set	179
5.4	ACG^* functions	187
5.5	Controlled convergence	190
5.6	Exercise	197
6	Integration in Several Dimensions	202
6.1	Introduction	202
6.1.1	Sets in \mathbb{R}^n	203
6.2	Divisions, partitions	204
6.3	The definition	210
6.4	Basic theorems	215
6.4.1	Prelude to Fubini's theorem	217
6.5	Other theorems in \mathbb{R}^n	220
6.5.1	Negligible sets	220
6.5.2	Henstock's lemma	220

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

<i>Contents</i>	vii
6.5.3 Absolute integrability	221
6.5.4 Convergence, measurability, AC	223
6.6 The Fubini theorem	229
6.7 Change of variables	236
6.7.1 Introductory examples	236
6.7.2 Notation, lemmas	240
6.7.3 The theorem	241
6.8 Exercises	248
7 Some Applications	252
7.1 Introduction	252
7.2 A line integral	253
7.2.1 Green's theorem	267
7.2.2 The Cauchy theorem	274
7.3 Differentiation of series	279
7.4 Dirichlet's problem and the Poisson integral	282
7.5 Summability of Fourier series	286
7.6 Fourier series and the space \mathcal{L}^2	289
7.7 Exercises	295
<i>Appendix 1</i> Supplements	299
<i>Bibliography</i>	305
<i>Index</i>	308

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

Preface

Presenting the theory of the integral to non-specialists is an old and everlasting problem. At most universities the Riemann integral is taught in introductory courses, even to future mathematicians. The reason for this is that the Riemann integral has an intuitive appeal and basic theorems are easy to prove. This, however, is all that can be said in its favour. This theory is not powerful enough for applications and when it comes to deeper results they are not any easier to prove than the corresponding results in more modern theories. It is true that Riemann with his approach to integration advanced mathematics significantly but that was almost a century and a half ago. We feel the time is now ripe to start teaching more comprehensive theories of integration at all levels.

The theory of integration employed by professional mathematicians was created by Henri Lebesgue at the beginning of the twentieth century. It could hardly be criticized and the mathematical community is happy with it. Unfortunately experience shows that, perhaps because of its abstract character, it is deemed to be difficult by beginners and non-mathematicians. It is not popular with physicists and engineers. The Lebesgue theory does not cover non-absolutely convergent integrals and there is a need then to consider improper integrals. It is an additional and important advantage of the theory expounded in this book that it includes all improper integrals.

In 1957 Jaroslav Kurzweil gave a new definition of the integral, which in some respects is more general than Lebesgue's. Ralph Henstock developed the theory further and started to advocate its use at the elementary level. The Kurzweil–Henstock theory preserves the intuitive appeal of the Riemann definition but has the power of the Lebesgue theory. The aim of this book is to present the Kurzweil–Henstock theory. We wish to give this powerful tool to non-mathematicians and under-

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)*Preface*

ix

graduates and we advocate the widest possible use of **one** integral at all levels. We believe that the desirability of teaching one integral at all levels was also part of the motivation for R. Henstock to develop the theory.

Both authors have taught the Kurzweil–Henstock integral at various levels and various universities, first of all at our home institutions, the National Institute of Education in Singapore and University of Queensland and also at Universität Erlangen–Nürnberg, the University of Canterbury, Northwest Normal University in Lanzhou and the University of the Philippines. We express our gratitude to the Mathematics Departments of these institutions for their understanding of our desire to teach a ‘new’ integral and support of our research. Our experience is positive at all levels and in the introductory courses, once the students grasped the concept of δ -fine partitions, they found the theory as easy as, or perhaps one should say no more difficult than, the Riemann theory.

Several books have appeared since the inception of the Kurzweil–Henstock theory. Most of these aim at the advanced or graduate level. This is so with the books which the inventors themselves wrote, [15], [16], [18] and [21]. Other books at the same level are Gordon’s [12], Pfeffer’s [37] and Lee’s [23]. The book by DePree and Swartz [8] does contain an introduction to Kurzweil–Henstock theory, but we in contrast cover more material and concentrate solely on integration. J. Mawhin’s *Introduction à l’Analyse* [28] contains the Kurzweil–Henstock integral; obviously it is in French. The book by McLeod [30] is closest to us in its spirit but we use very different and more systematic notation, which we feel is important at the elementary level. We also consider some topics in greater detail, relate the KH-integral to other integrals and give a range of applications including Fourier series.

We hope that our book will be useful at various levels. The first section of Chapter 1 and Chapter 2, with perhaps some omissions, can serve as a first (serious) course on integration. Later sections of Chapter 1 contain a fairly complete account of the Riemann integral but require more mathematical maturity and are **not** intended for a beginner or a non-mathematician. To indicate that these sections are not meant for the first reading they are typeset in a smaller font. We have expounded the Riemann theory to provide easily available comparison for someone who desires it. For instance, the non-integrable derivative of Example 1.4.5 gives an opportunity to appreciate the Fundamental Theorem 2.6.2 but it is far more difficult than the proof of the Fundamental Theorem itself. Chapters 3 and 6 together with some topics from Chapter 7 can

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

x

Preface

form the basis of a course which could be given instead of a first course in Lebesgue theory. Chapters 4 and 5 are not elementary; they give the most general convergence theorems for the Kurzweil–Henstock integral. Exercises are provided at the end of each Chapter. Exercises containing additional information which is worth reading even if one does not intend to work them out in detail are marked by ☺; exercises which are not easy are marked by ☹.

Finally we wish to acknowledge help when writing this book. We thank the editor of this series, John Loxton, for his friendly attitude and invaluable advice. We are grateful to David Tranah and particularly to Roger Astley from CUP for the care and expertise with which they have published our work. In writing we had advice on computer typesetting and presentation from our friends and colleagues. We specifically mention Anthony Miller from CSIRO in Adelaide, Ding Chuan Song from the Northwest Normal University in China, Chew Tuan Seng from the National University of Singapore, and Peter Adams, Keith Matthews and Ken Smith from the University of Queensland. Peter Adams also produced all figures in this book.

January 1999

Lee Peng Yee
Rudolf Vyborný

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

List of Symbols

Symbol	Description	Page
$\{(x, I)\}$	partition	4
$\{(x_k, I_k)\}$	partition	4
$\{(\xi, [u, v])\}$	partition	4
$\{(\xi_k, [u_k, v_k])\}$	partition	4
π	denotes partition	4
•	end of proof	
▪	filling an n -dimensional object	203
j	complex unit	69
\mathbb{C}	complex numbers	69
\mathbb{N}	positive integers	3
\mathbb{Q}	rationals	3
\mathbb{R}	reals	3
$\overline{\mathbb{R}}$	extended reals	64
$\overline{\mathbb{R}}^n$	see subsection 6.1.1	203
\mathbb{R}_+	positive reals	3
\mathbb{Z}	integers	3
f_{-1}	the inverse function	3
$\sup\{f; M\}$	l.u.b. of f over M	3
$\inf\{f; M\}$	g.l.b. of f over M	3
$ I $	length or content of I	3, 204
$\mathbf{1}_S$	characteristic function of S	3
$\mathcal{N} \int_a^b f$	Newton integral	1
$\mathcal{KH} \int_a^b f$	Kurzweil-Henstock integral	29
$\mathcal{L} \int_S f$	Lebesgue integral	137
$\int_\varphi f$	line integral	253

Cambridge University Press

978-0-521-77968-5 - Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee and Rudolf Vyborny

Frontmatter

[More information](#)

xii

List of Symbols

Symbol	Description	Page
$\mathcal{F} \int_a^b f$	Riesz integral	139
$\mathcal{M} \int_A^B f$	McShane integral	29
$\mathcal{SL} \int_A^B f$	SL-integral	154
$(A) \sum_1^\infty a_n$	Abel sum	286
$\sum_\pi f$	Riemann sum	4
$\sum_\pi f(\xi)(v - u)$	Riemann sum	4
$\sum_\pi f(y) I $	Riemann sum	4
$\sum_\pi \overset{c}{a} f$ or $\sum \overset{c}{a} f$	Riemann sum	37
$S(D)$	Darboux upper sum	6
$s(D)$	Darboux lower sum	6
$\pi \ll \delta$	π is δ -fine	23
$\Re z$	real part of the complex number z	69
$\Im z$	imaginary part of the complex number z	69
$\text{Var}_a^b F$	variation of F on $[a, b]$	84
f^N	truncated function	92
$m(S)$	measure of S	120
$m_n(S)$	measure of S in \mathbb{R}^n	228
$A \Delta B$	symmetric difference	122
\mathcal{E}	a function with small Riemann sums	134, 223
AC^*	see Definition 5.3.1	179
AC	see Definition 5.3.4	182
VB^*	see Definition 5.3.7	183
ACG^*	see Definition 5.4.1	187
ACG	see Definition 5.4.3	188
VBG^*	see Definition 5.4.4	188
$ x _u$	maximum norm of x in $\overline{\mathbb{R}^n}$	204
$ x _2$	Euclidean norm of x in $\overline{\mathbb{R}^n}$	204
$\mathcal{C}(a, h)$	cube centred at a	204