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Integral: An Easy Approach after Kurzweil and Henstock

Lee Peng Yee

The National Institute of Education, Singapore

Rudolf Výborný

The University of Queensland

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Preface

Presenting the theory of the integral to non-specialists is an old and everlasting problem. At most universities the Riemann integral is taught in introductory courses, even to future mathematicians. The reason for this is that the Riemann integral has an intuitive appeal and basic theorems are easy to prove. This, however, is all that can be said in its favour. This theory is not powerful enough for applications and when it comes to deeper results they are not any easier to prove than the corresponding results in more modern theories. It is true that Riemann with his approach to integration advanced mathematics significantly but that was almost a century and a half ago. We feel the time is now ripe to start teaching more comprehensive theories of integration at all levels.

The theory of integration employed by professional mathematicians was created by Henri Lebesgue at the beginning of the twentieth century. It could hardly be criticized and the mathematical community is happy with it. Unfortunately experience shows that, perhaps because of its abstract character, it is deemed to be difficult by beginners and non-mathematicians. It is not popular with physicists and engineers. The Lebesgue theory does not cover non-absolutely convergent integrals and there is a need then to consider improper integrals. It is an additional and important advantage of the theory expounded in this book that it includes all improper integrals.

In 1957 Jaroslav Kurzweil gave a new definition of the integral, which in some respects is more general than Lebesgue's. Ralph Henstock developped the theory further and started to advocate its use at the elementary level. The Kurzweil–Henstock theory preserves the intuitive appeal of the Riemann definition but has the power of the Lebesgue theory. The aim of this book is to present the Kurzweil–Henstock theory. We wish to give this powerful tool to non-mathematicians and under-

graduates and we advocate the widest possible use of **one** *integral at all levels. We believe that the desirability of teaching one integral at all levels was also part of the motivation for R. Henstock to develop the theory.*

Both authors have taught the Kurzweil–Henstock integral at various levels and various universities, first of all at our home institutions, the National Institute of Education in Singapore and University of Queensland and also at Universität Erlangen-Nürnberg, the University of Canterbury, Northwest Normal University in Lanzhou and the University of the Philippines. We express our gratitude to the Mathematics Departments of these institutions for their understanding of our desire to teach a 'new' integral and support of our research. Our experience is positive at all levels and in the introductory courses, once the students grasped the concept of δ-fine partitions, they found the theory as easy as, or perhaps one should say no more difficult than, the Riemann theory.

Several books have appeared since the inception of the Kurzweil–Henstock theory. Most of these aim at the advanced or graduate level. This is so with the books which the inventors themselves wrote, [15], [16], [18] and [21]. Other books at the same level are Gordon's [12], Pfeffer's [37] and Lee's [23]. The book by DePree and Swartz [8] does contain an introduction to Kurzweil–Henstock theory, but we in contrast cover more material and concentrate solely on integration. J. Mawhin's Introduction à l'Analyse [28] contains the Kurzweil–Henstock integral; obviously it is in French. The book by McLeod [30] is closest to us in its spirit but we use very different and more systematic notation, which we feel is important at the elementary level. We also consider some topics in greater detail, relate the KH-integral to other integrals and give a range of applications including Fourier series.

We hope that our book will be useful at various levels. The first section of Chapter 1 and Chapter 2, with perhaps some omissions, can serve as a first (serious) course on integration. Later sections of Chapter 1 contain a fairly complete account of the Riemann integral but require more mathematical maturity and are **not** *intended for a beginner or a non-mathematician. To indicate that these sections are not meant for the first reading they are typeset in a smaller font. We have expounded the Riemann theory to provide easily available comparison for someone who desires it. For instance, the non-integrable derivative of Example 1.4.5 gives an opportunity to appreciate the Fundamental Theorem 2.6.2 but it is far more difficult than the proof of the Fundamental Theorem itself. Chapters 3 and 6 together with some topics from Chapter 7 can* *form the basis of a course which could be given instead of a first course in Lebesgue theory. Chapters 4 and 5 are not elementary; they give the most general convergence theorems for the Kurzweil–Henstock integral. Exercises are provided at the end of each Chapter. Exercises containing additional information which is worth reading even if one does not intend to work them out in detail are marked by ⓘ; exercises which are not easy are marked by ⓘ .*

Finally we wish to acknowledge help when writing this book. We thank the editor of this series, John Loxton, for his friendly attitude and invaluable advice. We are grateful to David Tranah and particularly to Roger Astley from CUP for the care and expertise with which they have published our work. In writing we had advice on computer typesetting and presentation from our friends and colleagues. We specifically mention Anthony Miller from CSIRO in Adelaide, Ding Chuan Song from the Northwest Normal University in China, Chew Tuan Seng from the National University of Singapore, and Peter Adams, Keith Matthews and Ken Smith from the University of Queensland. Peter Adams also produced all figures in this book.

January 1999 *Lee Peng Yee*
Rudolf Výborný

List of Symbols

Symbol	Description	Page
$\{(x, I)\}$	partition	4
$\{(x_k, I_k)\}$	partition	4
$\{(\xi, [u, v])\}$	partition	4
$\{(\xi_k, [u_k, v_k])\}$	partition	4
π	denotes partition	4
$\bullet$	end of proof	
$\blacksquare$	filling an n-dimensional object	203
$\jmath$	complex unit	69
$\mathbb{C}$	complex numbers	69
$\mathbb{N}$	positive integers	3
$\mathbb{Q}$	rationals	3
$\mathbb{R}$	reals	3
$\overline{\mathbb{R}}$	extended reals	64
$\overline{\mathbb{R}}^n$	see subsection 6.1.1	203
$\mathbb{R}_+$	positive reals	3
$\mathbb{Z}$	integers	3
f_{-1}	the inverse function	3
$\sup\{f;\ M\}$	l.u.b. of f over M	3
$\inf\{f;\ M\}$	g.l.b. of f over M	3
$\|I\|$	length or content of I	3, 204
$\mathbf{1}_S$	characteristic function of S	3
$\mathcal{N}\int_a^b f$	Newton integral	1
$\mathcal{KH}\int_a^b f$	Kurzweil-Henstock integral	29
$\mathcal{L}\int_S f$	Lebesgue integral	137
$\int_\varphi f$	line integral	253

Symbol	Description	Page
$\mathcal{F}\int_a^b f$	Riesz integral	139
$\mathcal{M}\int_A^B f$	McShane integral	29
$\mathcal{SL}\int_A^B f$	SL-integral	154
$(A)\sum_1^\infty a_n$	Abel sum	286
$\sum_\pi f$	Riemann sum	4
$\sum_\pi f(\xi)(v-u)$	Riemann sum	4
$\sum_\pi f(y)\|I\|$	Riemann sum	4
$\sum_\pi {}_a^c f$ or $\sum {}_a^c f$	Riemann sum	37
$S(D)$	Darboux upper sum	6
$s(D)$	Darboux lower sum	6
$\pi \ll \delta$	π is δ-fine	23
$\Re z$	real part of the complex number z	69
$\Im z$	imaginary part of the complex number z	69
$\mathrm{Var}_a^b F$	variation of F on $[a, b]$	84
f^N	truncated function	92
$m(S)$	measure of S	120
$m_n(S)$	measure of S in $\mathbb{R}^n$	228
$A \triangle B$	symmetric difference	122
$\mathcal{E}$	a function with small Riemann sums	134, 223
AC^*	see Definition 5.3.1	179
AC	see Definition 5.3.4	182
VB^*	see Definition 5.3.7	183
ACG^*	see Definition 5.4.1	187
ACG	see Definition 5.4.3	188
VBG^*	see Definition 5.4.4	188
$\|x\|_u$	maximum norm of x in $\overline{\mathbb{R}}^n$	204
$\|x\|_2$	Euclidean norm of x in $\overline{\mathbb{R}}^n$	204
$C(a,h)$	cube centred at a	204