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Singular Perturbations of Differential Operators

Solvable Schrödinger Type Operators

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Human thought, flying on the trapezes of the star-filled universe, with mathematics stretched beneath, was like an acrobat working with a net but suddenly noticing that in reality there is no net, and Martin envied those who attained that vertigo and, with a new calculation, overcame their fear.

V. V. Nabokov, Glory.

Человеческая мысль, летающая на трапециях звездной вселенной, с протянутой под ней математикой, похожа была на акробата, работающего с сеткой, но вдруг замечающего, что сетки, в сущности, нет, - и Мартын завидовал тем, кто доходит до этого головокружения и новой выкладкой превозмогает страх.

В. В. Набоков, Подвиг.

A Mielikki e Solvejg.
con tutto il cuore
e tanta gratitudine

Моим родителям
Борису Владимировичу и
Наталии Павловне

Preface

Singular perturbations of Schrödinger type operators are of interest in mathematics, e.g. to study spectral phenomena, and in applications of mathematics in various sciences, e.g. in physics, chemistry, biology, and in technology. They also often lead to models in quantum theory which are solvable in the sense that the spectral characteristics (eigenvalues, eigenfunctions, and scattering matrix) can be computed. Such models then allow us to grasp the essential features of interesting and complicated phenomena and serve as an orientation in handling more realistic situations.

In the last ten years two books have appeared on solvable models in quantum theory built using special singular perturbations of Schrödinger operators. The book by S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden [39] describes the models in rigorous mathematical terms. It gives a detailed analysis of perturbations of the Laplacian in $\mathbf{R}^d, d = 1, 2, 3$, by potentials with support on a discrete finite or infinite set of point sources (chosen in a deterministic, respectively, stochastic manner). Physically these operators describe the motion of a quantum mechanical particle moving under the action of a potential supported, e.g., by the points of a crystal lattice or a random solid. Such systems and models are also described in physical terms in the book by Yu.N.Demkov and V.N.Ostrovsky [255], which also contains a description of applications in other areas such as in optics and electromagnetism. Let us also remark that a translation of the book [39] in Russian has been published with additional comments and literature [40]. Since the appearance of [39, 40, 255] several important new developments have taken place. It is the main aim of the present book to present some of these new developments in a unified formalism which also puts some of the basic results of the preceding books into a new light. The new developments concern in particular a systematic study of finite rank perturbations of (self–adjoint) operators (in particular differential operators), of generalized (singular) perturbations, of the corresponding scattering theory as well as infinite rank perturbations and multiple particles (many–body) problems in

quantum theory. We also present the theory of point interaction Hamiltonians, as a particular case of a general theory of singular perturbations of differential operators. This theory has received steadily increasing attention over the years also for its many applications in physics (solid state physics, nuclear physics), electromagnetism (antennas), and technology (metallurgy, nanophysics).

We hope this monograph can serve as a basis for orientation in a rapidly developing area of analysis, mathematical physics and their applications.

October 1998

S.Albeverio (Bonn), P.Kurasov (Stockholm)

Acknowledgments

Work on this monograph has ranged over several years and we are grateful to many persons and institutions who helped us to accomplish it.

We enjoyed the collaboration with many mathematicians and physicists concerning topics discussed in this book. First of all we acknowledge our great indebtedness to the late Raphael Høegh-Krohn, whose original inspiration was essential to our whole project. Special thanks also go to Boris Pavlov and Ludwig Faddeev, who supported the project at all stages. We would also like to thank

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Material included in the monograph has been discussed at several international conferences and symposia held in different scientific centres all over the world. In particular it was presented in a series of lectures given by the second named author at the international workshops 'Spectral analysis of some second differential and difference operators' held at the Banach Center, Warsaw, in August-September 1997 and 1998 (organized by J.Janas and S.Naboko). We would like to thank the Banach Center for their hospitality. Another course of lectures on the material included was given in January 1996 by the same author at the Department of Mathematical and Computational Physics of St.Petersburg University.

Other presentations of material in this monograph have been given at the 'SISSA Workshops on singular Schrödinger operators' in September 1994 and 'Semiclassical Limit of Quantum Mechanics and Non-linear Schrödinger equation' in July 1998 (organized by G.F. Dell'Antonio, R. Figari and A. Teta), the Oberwolfach meeting on 'Spectral Theory' in July 1998 (organized by M. Van den Bergh, J.A. Van Casteren, M.Demuth) and on the Bad Herrenalb Conference 'Evolution equations and their applications in physical and life sciences', in September 1998 (organized by G. Lumer and L. Weis).

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