Introduction: Geometric Approaches to Differential Equations*

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The year 1997 marked the 30th anniversary of the discovery of the inverse scattering transform, a technique for solving the initial value problem for certain nonlinear partial differential equations of mathematical physics [GGKM]. This event opened the flood gate for a large amount of intriguing research and has had a powerful effect in large parts of mathematics and physics ranging from algebraic geometry to optical fibres (see, for example, [AC]; [AN]; [DEGM]; [Pal]). One of these effects has been to revive the study of the *geometric* aspects differential equations, owing to the discovery that the inverse scattering transform has significant geometric properties.

The study of the geometric properties of differential equations goes back at least to the work of Monge on the minimal surface equation. Monge's investigations were greatly extended and codified in one of the great works of late 19th century mathematics, Darboux's "*Leçons sur la Théorie de Surfaces*" [Da] which is an extensive study of the geometry of surfaces in Euclidean space. However, as Robert Hermann has remarked, much of the four-volume work is devoted to various topics in partial differential equations, such as Laplace transformations which occupies much of volume 2.

Thus, from the very beginning there has been a fruitful correspondence between differential geometry on the one hand and differential equations on the other. Questions in differential geometry could be solved, or at least illuminated, by studying the differential equations that were implied by the geometry. Conversely, questions in differential equations benefit by a study of the accompanying geometric setting.

To give an illustrative example, we briefly describe the classical Laplace transformation and some of its more recent generalisations. The Laplace transformation (unrelated to the Laplace transform of harmonic analysis) is a transformation for surfaces in $\mathbf{E}^3$, three-dimensional Euclidean space. Suppose a surface S in $\mathbf{E}^3$ has a net of curves conjugate for the second fundamental form. Then a parametrisation $X(u,v)$ of S satisfies a second-order hyperbolic equation

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v \tag{1}$$

where $\Gamma^i_{12}, i = 1,2$, are the Christofel symbols for the surface S. Fix the coordinate v, say v_0, and consider the ruled surface $Y(t,u;v_0) = X(u,v_0) + tX_v(u,v_0)$, where subscript v denotes partial differentiation. The surface Y determines a curve, the edge of regression (see [Sp], p.208). It follows that, for each u, there

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is a parameter value $t = t_0$ such that $Y(t_0, u; v_0)$ is on the edge of regression (see [KT95]); this, in turn, defines a point in $\mathbf{E}^3$. As v_0 varies, we obtain, in general, a new surface parametrised by $X_1(u, v)$ with the remarkable property that the net of curves u, v are still conjugate for the second fundamental form and hence X_1 satisfies a partial differential equation of the same form as equation (1). The new surface S_1 parametrised by X_1 is one of the two possible Laplace transformations of S. The other one is obtained by reversing the role of u and v in the above construction.

Now, it can happen that the image of a Laplace transformation, say S_1, can be degenerate forming a curve rather than a surface, as, for example, when S is a surface of rotation (see [KT95]). In this case, it turns out that the corresponding partial differential equation is solvable by quadrature. This fact, together with the general Laplace transformations described above, leads to an integration procedure and general "transformation theory" for linear hyperbolic equations of the form

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0.$$

This classical construction has been extended in a number of ways in the more recent literature. In [Ch], Chern has given a geometric description of a generalisation of the Laplace transformation to a class of n-dimensional submanifolds of projective space which had previously been studied by Cartan. These submanifolds, which Chern calls Cartan submanifolds, admit a parametrisation by a conjugate net. For each n-dimensional Cartan submanifold, Chern constructs $n(n-1)$ generalised Laplace transformations which, generically, define n-dimensional Cartan submanifolds. In [KT96], Kamran and Tenenblat consider Cartan submanifolds in Euclidean space. They give a generalisation of the classical Laplace transformations and the classical integration procedure alluded to above. It transpires that the systems of partial differential equations that Kamran and Tenenblat thereby construct find application in the study of the conserved quantities for semi-Hamiltonian, strongly hyperbolic systems of hydrodynamic type [Ts]. For the reader wishing to pursue these intriguing matters further, the papers by Kamran and Tenenblat referenced above are highly recommended as is the book [Ten] which may be regarded as a companion to the present volume. Finally, in [Va], it is shown how a consideration of the theory of Laplace transformations leads to an integration procedure for a generalised Toda field theory.

Thus, the example of the classical Laplace transformation together with its recent generalisations and applications provide a nice illustration of the above mentioned correspondence between geometry and differential equations. This interplay is one of the chief themes of this book.

Indeed, it is amply illustrated in the contribution to this volume by Colin Rogers, Wolfgang Schief and Mark Johnston who present an account of the role of geometry to the theory of completely integrable systems. In the classical literature, a *completely integrable system* is a system of ordinary differential equations for an even number of dependent variables, say $2n$, which has a Hamiltonian structure and possesses n constants of the motion which pairwise commute with respect to the natural Poisson bracket. For more details on this, the reader is referred to Geoff Prince's contribution to this volume and to the references therein. The discovery of the inverse scattering transform in 1967 introduced the notion of an *infinite-dimensional* Hamiltonian system with in-finitely many "constants of the motion" which pairwise commute with respect to an appropriate Poisson bracket. Thus, a theory has developed generalising the classical theory of completely integrable systems such as may be found, for example, in classical texts such as Whittacker [Wh]. The many beautiful and intricate properties of infinite dimensional completely integrable systems are far too numerous to be reviewed here. For an account of many of these matters, the reader is directed to the contribution of Annalisa Calini to this volume. In addition, the recent paper [Pal] by R. S. Palais and the book [AN] by A. C. Newell are also highly recommended.

Nevertheless, without going into all the details here, let us observe that if an equation is solvable by the inverse scattering transform, such as, for example, the celebrated Korteweg–de Vries equation (KdV)

$$u_t + uu_x + u_{xxx} = 0, \tag{2}$$

then fortuitously the equation has many other properties which are what Newell refers to as the "miracles of soliton mathematics". One of these miracles is a "transformation" discovered by the Swedish mathematical physicist and geometer A.V. Bäcklund (1845–1922) and these days referred to as a *Bäcklund transformation*. The reason for the inverted commas is that one does not really have a transformation in the sense of a locally defined diffeomorphism on a manifold but rather a "correspondence" among the solutions of a given equation or between the solutions of two distinct equations. Bäcklund's original discovery concerned another famous equation from the theory of solitons, the sine-Gordon equation

$$u_{xy} = \sin u, \tag{3}$$

which arises naturally within the theory of surfaces in three-dimensional Euclidean space. Bäcklund discovered the remarkable fact that if u is any solution of equation (3) then u' defined by the first order system

$$\begin{aligned} (\frac{u'-u}{2})_x &= \lambda \sin(\frac{u'+u}{2}) \\ (\frac{u'+u}{2})_y &= \frac{1}{\lambda} \sin(\frac{u'-u}{2}) \end{aligned} \tag{4}$$

is also a solution of equation (3). Here λ is an arbitrary real parameter. Similarly, if u' is a solution of (3) then system (4) provides another solution of (3). For instance, if one takes for u the zero solution of the sine-Gordon equation, one may easily solve the resulting equations (4) to obtain the so-called 1-soliton solution, that is a "stable" travelling wave solution which has remarkable "elastic" properties under interaction with other solitons.

A powerful consequence of this *Bäcklund correspondence* is the so-called Bianchi Permutability Theorem whereby, given three distinct solutions of the sine-Gordon equation (3), a fourth solution may be constructed by purely algebraic means. This phenomenon, referred to as a nonlinear superposition formula, and the associated Bäcklund transformation, appears only to occur for a very privileged class of nonlinear partial differential equations. Nevertheless, it is observed that this phenomenon always accompanies any equation that is solvable by the inverse scattering transform. In this regard we remark that the

Bäcklund transformation for the sine-Gordon equation has been known for over 100 years and hence its discovery considerably predates the inverse scattering transform. However, the Bäcklund transformation for the KdV equation was discovered by Wahlquist and Estabrook in 1973 [WE], approximately five years after the discovery of the inverse scattering transform. Since this time, Bäcklund transformations have been discovered for all the equations known to be solvable by the inverse scattering transform. The deeper understanding of Bäcklund transformations and similar phenomena within the theory of completely integrable systems are outstanding problems of great importance.

The chapter by Rogers, Schief and Johnston explores and reviews many of these ideas giving a clear introduction to a number of recent results which illustrate the role played by geometry in differential equations. By way of introducing the subject, the authors begin by studying the Bäcklund transformation for the sine-Gordon equation. Their derivation is particularly interesting in that it is shown to arise explicitly from a simple geometric construction. Beginning with the Gauss equations for the parametrisation of a surface in $\mathbf{E}^3$ and their compatibility conditions, the Mainardi–Codazzi equations and the Gauss formula, they show that making a geometrically motivated choice of new dependent and independent variables leads to the so-called *Bianchi system* for surfaces of negative Gauss curvature. In the special case of constant negative Gauss curvature, the Bianchi system reduces to the sine-Gordon equation. It follows that each solution of the sine-Gordon equation then leads, via the Gauss equations, to a surface of constant negative Gauss curvature (a pseudospherical surface). Given a pseudospherical surface in $\mathbf{E}^3$, a simple geometric construction enables the authors to construct another pseudospherical surface. The analytic counterpart of this geometric construction, via the Gauss-Weingarten equations, leads to the sine-Gordon Bäcklund transformation.

Here again, as in the work of Chern, and of Kamran and Tenenblat mentioned above, we see that a nontrivial fact within theory of differential equations has a simple counterpart in differential geometry. Rogers, Schief and Johnston go on to discuss a number of other interesting topics within the general theme of Bäcklund transformations. For instance, they produce graphs of pseudospherical surfaces arising from "breather solutions" of the sine-Gordon equation. As a natural development of these ideas the *motion* of pseudospherical surfaces is considered. A host of topics are then discussed ending with a striking connection with the work of C. Loewner on the equations of compressible gas flow and the so-called "infinitesimal" Bäcklund transformations leading to new completely integrable systems in more than two independent variables. It is clear from the chapter by Rogers, Schief and Johnston that Bäcklund transformations form a deep and compelling topic within both differential geometry and differential equations.

The theme of completely integrable systems and differential geometry continues in the contribution of Annalisa Calini. Here, the simplest of geometric objects, curves, play a fundamental role. The objects of interest are *vortex filaments*, approximately one-dimensional regions in a fluid where the velocity distribution has a rotational component. Smoke rings are a common example of this phenomenon. Modeling vortex filaments as closed curves in $\mathbb{R}^3$ leads to the vortex filament equation which is shown to have a Hamiltonian formulation on an infinite-dimensional phase space (the loop space). Remarkably, it transpires that if a closed curve evolves according to the vortex filament equation then a function expressed in terms of the curvature and torsion of the curve evolves in accordance with the cubic nonlinear Schrodinger equation (NLS), a well -known soliton equation. It follows that the vortex filament equation is completely integrable. A further link is made with a differentiated version of the vortex filament equation leading to an evolution equation for the tangent vector to the closed curve identified as the continuous Heisenberg model (HM). It has been known for some time that the HM and NLS equations are related by a gauge transformation of associated "linear problems", commonly referred to as Lax pairs within the theory of completely integrable systems. Among the beautiful results of this chapter is a geometric interpretation of this gauge transformation in terms of a canonical connection on the circle bundle of S^2. In addition, use is made of algebraic geometry to obtain solutions of the HM and the construction of associated closed curves in $\mathbb{R}^3$. The paper concludes with a study of Bäcklund transformations and immersed knots and a discussion of future directions in the field.

Another thread that runs through the geometric aspects of differential equations is the notion of symmetry. Here the contribution of Sophus Lie is decisive and all pervading. A *symmetry* of a differential equation is a transformation that maps solutions of the equation to other solutions. Lie discovered that the collection of all such transformations form what is now called a Lie transformation group. Gradually, the notion of a transformation group was refined and it became clear, with the work of Élie Cartan and others that there is an abstract structure underlying transformation groups, namely Lie groups. From the very beginning, a distinction arose between the finite groups (those whose elements are parametrised by finitely many real or complex numbers) and the infinite groups whose elements depend upon arbitrary functions. The abstract structure for Lie groups mentioned above applies to the finite case. The construction of an abstract structure in the infinite case is still largely an open problem. In the infinite case, these transformation groups have come to be called *infinite Lie pseudogroups*. The qualifier 'pseudo' comes from the fact that the elements of the 'group' do not form a group in the usual sense since composition between two arbitrary elements may not be defined.

Many differential equations of interest and indeed many other 'geometric objects' have symmetries which are precisely infinite Lie pseudogroups. Hence, the structure of these groups is of great practical and theoretical importance. For example, one of Lie's results states that if an ordinary differential equation of order k admits a symmetry Lie group which is *solvable* and has dimension k, then the construction of solutions of the equation may be reduced to quadrature (see Bluman and Kumei [BK]; Olver [O]). Lie's result underscores one of the most important aspects of the geometry of differential equations, namely, structural information on admitted symmetry groups yields significant results about the differential equations themselves. This may be compared with the role played by the Galois group in the study of polynomial equations.

Thus, historically, we see two geometric influences impinging upon differential equations. On the one hand, the role of the classical differential geometry of surfaces, mentioned above, and on the other, the development of the notion of symmetry group leading to a *geometry* for differential equations. But what does it mean to talk about a "geometry" for differential equations? In the current state of development a definitive answer to this question is still under construction. In their expository article "Towards a geometry of differential equations", Bryant, Griffiths and Hsu [BGH] have been content to confine themselves more or less to the description of significant examples within the field. This is surely the path of wisdom and their article is highly recommended for anyone interested in the subject. However, a few broad features can be discerned which may help the reader interpret these words. Firstly, let's turn to a setting in which the word "geometry" has a well-understood meaning. A Riemannian manifold is a differentiable manifold M together with a smoothly varying inner product g on each tangent space. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are *locally equivalent* if and only if there is a local diffeomorphism $\phi : M_1 \to M_2$ which identifies their Riemannian structures: $\phi^* g_2 = g_1$, where superscript $*$ denotes pullback.

Riemannian manifolds M_1 and M_2 are then said to be *locally isometric*. One knows, for example, that a necessary condition for a pair of two-dimensional Riemannian manifolds (surfaces) to be locally isometric is that their Gauss curvatures be equal: $\phi^* K_2 = K_1$. One can say that the broad goal of Riemannian geometry is to describe all Riemannian manifolds up to local isometry (the natural equivalence for this class of geometric objects). In the course of trying to answer this question, one discovers the Riemann curvature tensor and other invariants that label the equivalence classes within this set of objects. Of course, the problem as stated is impossible to solve in any meaningful way but it sets down the tone of the subject. It turns out that one can study differential equations in much the same spirit as one studies Riemannian manifolds. The basic venue is the k^{th} order jet bundle of maps of $\mathbb{R}^m \to \mathbb{R}^n$, $J^k(\mathbb{R}^m, \mathbb{R}^n)$. In order to describe this, we will pause briefly to set up the basic structures of the subject. So as to make rapid progress, a somewhat elementary description of jet bundles and the associated contact structure will be given. The reader may consult [BC3G] and [KV] for more details. The last reference is a noteworthy attempt to present the modern geometry of differential equations in a concise and accessible form. Since most of the considerations of this book shall be local, no generality is lost for the present purposes by taking our manifolds to be open subsets of $\mathbb{R}^n$. We wish ultimately to study systems of differential equations in p independent variables $x_1, x_2, \ldots, x_p$ and q dependent variables $u^1, u^2, \ldots, u^q$. Denote by X the space of independent variables and by U the space of dependent variables and consider the trivial bundle $\pi : X \times U \to X$. Two smooth sections f and g of π are said to be equivalent to order k at x if and only if

$$\partial_{x_I}^{|I|} f(x) = \partial_{x_I}^{|I|} g(x), \; x_I = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_p^{\alpha_p}$$

where I is a multi-index of order less than or equal to k, that is, $|I| = \alpha_1 + \alpha_2 + \cdots + \alpha_p \leq k$. In other words, two smooth sections are equivalent to order k at x if their Taylor coefficients agree up to order k at x. Lie called the equivalence class of a smooth section f at a point x a contact element of order k. The modern terminology for this is the *k-jet of f at x* and is denoted $j_x^k f$. The set of all k-jets of smooth sections of π constitutes the bundle of k-jets of maps $X \to U$ and denoted $J^k(X, U)$. The set $J^k(X, U)$ can be given the structure of a differentiable manifold. We introduce local coordinates $(x_i, u^\alpha, u_{i_1}^\alpha, u_{i_1 i_2}^\alpha, \ldots, u_{i_1 i_2 \ldots i_k}^\alpha)$, where the $u_{i_1 i_2 \ldots}^\alpha$ are symmetric in the lower indices and $i_1, i_2, \ldots = 1, 2, \ldots, p$ and such that

$$x_i(j_x^k f) = x_i, \quad u^\alpha(j_x^k f) = f^\alpha(x),$$
$$u_I^\alpha(j_x^k f) = \partial_{x_I}^{|I|} f^\alpha(x),$$

The zeroth order jet bundle, $J^0(X,U)$, is identified with $X \times U$. The k-graph of a smooth section f of π is the map

$$j^k f : X \to J^k(X,U)$$

defined by

$$x \mapsto j^k_x f.$$

Thus, the image of the k-graph of a section is a p-dimensional immersed submanifold of $J^k(X,U)$. Each jet bundle $J^k(X,U)$ comes equipped with a module of differential 1-forms spanned by

$$\theta^\alpha_I = du^\alpha_I - \sum_{i=1}^{p} u^\alpha_{I,i} dx_i$$

where I is a multi-index of order less than or equal to $k-1$. This module is denoted $\Omega^k(X,U)$ or simply Ω^k and called the *contact system* on J^k.

Example 1. $\Omega^1(\mathbb{R},\mathbb{R}) = \{du - u_1 dx_1\}$, $\Omega^2(\mathbb{R}^2,\mathbb{R}) = \{du - u_1 dx_1 - u_2 dx_2, du_1 - u_{11} dx_1 - u_{12} dx_2, du_2 - u_{21} dx_1 - u_{22} dx_2\}$.

The contact system on J^k is an example of an *exterior differential system*, generating an ideal in the ring of differential forms on a differentiable manifold. Since this particular differential system is generated by 1-forms, it is also called a *Pfaff system*. As Niky Kamran describes in greater detail in his contribution to this volume, for a given differential system E on a manifold M, generated by 1-forms, 2-forms, ... there may be submanifolds $N \subset M$ where all the forms in E are zero when pulled back to N. That is, if $i : N \to M$ is the natural inclusion, then $i^* E = 0$. Such a submanifold is called an *integral submanifold*. Another way to view this is that the differential system annihilates each tangent space to the submanifold N. Thus, N is an integral submanifold of E if and only if $v_p \in \ker(E_p), \forall v_p \in T_p N, p \in N$. Note that the image of the k-graph of any function $f : X \to U$ is an integral submanifold of the k^{th}-order contact system, $\Omega^k(X,U)$.

We can now say how these constructions are related to differential equations. We begin by giving some illustrative examples. In view of the definition of the jet bundle, we will think of a differential equation as a relationship between the coordinates of an appropriate jet bundle. Such a relationship defines a subset of the jet bundle. Indeed, we shall specialise this somewhat and consider differential equations which define embedded submanifolds.

Example 2. First order ordinary differential equation, $y' = F(x,y)$. Consider the embedded submanifold of $J^1(\mathbb{R},\mathbb{R})$ defined by

$$\mathcal{R} = \{(x,u,u_1) \in J^1(\mathbb{R},\mathbb{R}) \mid u_1 - F(x,u) = 0\}.$$

The manifold $\mathcal{R}$ is a surface in J^1 which is also a graph over the xu-plane with local coordinates x,u. A curve $(x,y(x))$ in the xu-plane defines a solution of the differential equation $y' = F(x,y)$ if it "lifts to $\mathcal{R}$" under the 1-graph $x \mapsto (x,y(x),y'(x))$. That is, $y(x)$ is a solution of the differential equation if and only if the image of the 1-graph of $y(x)$ is a curve whose height above each point $(x,y(x))$ in the xu-plane is $F(x,y)$. This, after all, is precisely what we mean when we say that a function $y(x)$ "satisfies" the differential equation. We can say this a little more precisely by introducing the natural inclusion

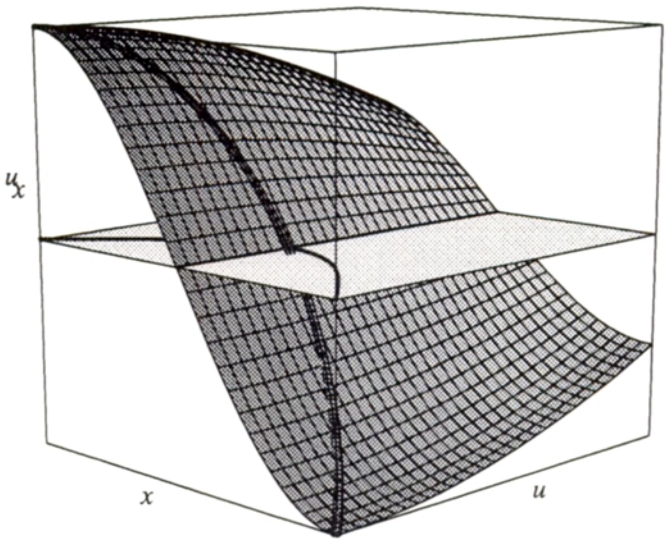

Figure 1: Submanifold $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R})$ with a solution of a first order ordinary differential equation and its lift under its 1-graph.

map $i : \mathcal{R} \to J^1(\mathbb{R}, \mathbb{R})$ and letting any curve $S \subset \mathcal{R}$ be the image of a map $h : X \to \mathcal{R}$. Then S defines a solution of the differential equation if and only if $i \circ h : X \to J^1(\mathbb{R}, \mathbb{R})$ is the 1-graph of a differentiable function. This will be the case if and only if $(i \circ h)^*\Omega^1(\mathbb{R}, \mathbb{R}) = 0$. Thus, the image of a map $h : X \to \mathcal{R}$ will define a solution of the differential equation if and only if it is an integral submanifold of the differential system $i^*\Omega^1$, the contact system restricted to the differential equation manifold $\mathcal{R}$. Thus, if we wish to study the solutions of a first order ordinary differential equation we can equivalently study the integral submanifolds of a certain differential system, namely the restricted contact system $i^*\Omega^1 = \{du - F(x, u)dx\}$.

Example 3. First order partial differential equation, $u_y = F(x, y, u, u_x)$. Once again, introduce the embedded submanifold

$$\mathcal{R} = \{(x_1, x_2, u, u_1, u_2) \in J^1(\mathbb{R}^2, \mathbb{R}) \mid u_2 - F(x_1, x_2, u, u_1) = 0\},$$

of $J^1(\mathbb{R}^2, \mathbb{R})$ and inclusion map i. As in the above example, a submanifold S in $\mathcal{R}$ will define a solution of the equation if and only if S is an integral submanifold of the restricted contact system $i^*\Omega^1(\mathbb{R}^2, \mathbb{R}) = \{du - u_1 dx_1 - F dx_2\}$. Here the pictorial analogy is harder to pursue since S is a two-dimensional submanifold of the four-dimensional manifold $\mathcal{R}$. However, the geometric situation is exactly the same. A surface in xyu-space defines a solution (in fact, it's the 0-graph of a solution) if and only if it "lifts to $\mathcal{R}$" under its 1-graph. In the case of a first order ordinary differential equation, this lift assigns the slope of the tangent

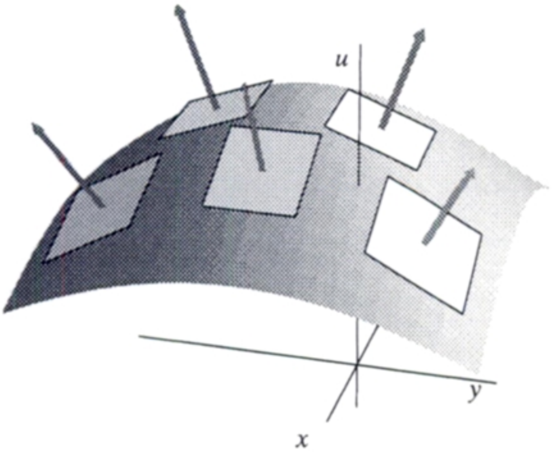

Figure 2: Graph of a solution of a first order partial differential equation showing the orientation of some tangent planes.

line at each point (x, u) of the curve to a length along the u_1-axis, forming a curve in a three-dimensional space. In the case of a first order partial differential equation, the lift assigns the value of u_x, u_y at each point (x, y, u) of a solution surface to the u_1u_2-plane, thereby specifying the orientation of each tangent plane. Thus, the 0-graph of a function is the 0-graph of a solution if and only if the orientation of each of its tangent planes is constrained by $\mathcal{R}$.

Example 4. Second order partial differential equation, $u_{yy} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy})$. The embedded submanifold is

$$\begin{aligned}\mathcal{R} &= \{(x_1, x_2, u, u_1, u_2, u_{11}, u_{12}, u_{22}) \in J^2(\mathbb{R}^2, \mathbb{R}) \mid \\ &\qquad u_{22} - F(x_1, x_2, u, u_1, u_2, u_{11}, u_{12}) = 0\},\end{aligned}$$

being a subset of $J^2(\mathbb{R}^2, \mathbb{R})$. As before if i denotes the inclusion, then the solutions of the second order partial differential equation may be studied by studying the integral submanifolds of the restricted contact system

$$i^*\Omega^2(\mathbb{R}^2, \mathbb{R}) = \{du - u_1dx_1 - u_2dx_2, du_1 - u_{11}dx_1 - u_{12}dx_2, du_2 - u_{21}dx_1 - Fdx_2\}.$$

Here the pictorial analogy is even harder to pursue than in the previous case since the solutions are two-dimensional submanifolds of a seven-dimensional manifold, $\mathcal{R}$. However, the geometric situation is similar. The 0-graph of a function $X \to U$ is the 0-graph of a solution of the second order partial differential equation if and only if its 2-graph lifts to $\mathcal{R}$. This gives a constraint on the orientation of the tangent planes (the 1-jets) as well as the nature of the "local quadric surfaces" determined by the second order derivatives at each point (the 2-jets).

It will now be clear to the reader how this geometric construction describes differential equations and their solutions. A subset of an appropriate jet bundle determines the "constraints" imposed on a function and its derivatives in order that the function be a solution. In the above examples, this subset is always taken to be a submanifold; in general this need not be the case. In addition, these submanifolds have defined upon them, in a canonical way, distinguished differential systems whose integral submanifolds are in correspondence with the solutions of the original systems of differential equations. These differential systems (the restricted contact systems) have useful properties that one may exploit to gain insight into the nature of the differential equations which one would not be able to gain simply by studying the differential equations themselves.

We are now able to complete the analogy, promised earlier, between Riemannian geometry and a "geometry of differential equations". A *differential equations manifold* is an embedded submanifold of an appropriate jet bundle, $\mathcal{R}$, together with its restricted contact structure, $\mathcal{C}$, obtained by restricting the canonical contact structure on J^k to $\mathcal{R}$. Two differential equations manifolds $(\mathcal{R}_1, \mathcal{C}_1)$ and $(\mathcal{R}_2, \mathcal{C}_2)$ are *locally equivalent* if and only if there is a diffeomorphism $\phi : \mathcal{R}_1 \to \mathcal{R}_2$ which identifies their contact structures: $\phi^* \mathcal{C}_2 = \mathcal{C}_1$.[1] This is nothing more than a geometric way of saying that two differential equations can be transformed, one into the other, by a change of dependent and independent variables. The association of the contact structure to the differential equation is the essential step in the "geometrisation". One can now study the pair $(\mathcal{R}, \mathcal{C})$ as a geometric object and seek its invariants in the same way as one studies Riemannian structures to uncover isometric invariants. The resulting class of transformations are called *contact transformations*. Two differential equations manifolds which are locally contact equivalent essentially describe the same differential equation. Thus, as in Riemannian geometry, one can set down the broad ultimate goal of the geometry of differential equations: *describe the local and global structure of differential equations manifolds and their integral submanifolds (solutions)*. Thus, much of the geometry of differential equations "reduces" to the problem of studying exterior differential systems.

Though the geometry of differential equations has a long history going back to the work of Monge, the subject has had its most significant impetus from the work of late 19th and early 20th century mathematicians such as G. Darboux (1842–1917), S. Lie (1842–1899), E. Goursat (1858–1936), W. Killing (1847–1923), A. Tresse (1868–1958), E. Cartan (1869–1951) and E. Vessiot (1865–1952), among others. The influence of Cartan to the modern formulation of the subject is the equal of Lie's. It was Cartan who introduced the theory of exterior differential systems to the study of various problems in differential equations and differential geometry. A review of Cartan's work is therefore of great importance in the history of the modern theory of the geometry of differential equations. In this regard, the book [AR] and paper [ChC], giving broad overviews of Cartan's life and work are helpful. In addition, because of its very significant implications for the geometric aspects of differential equations, we also mention the book by

[1]This is not the only type of exterior differential system that one may associate to a differential equation, however, it is a very important one, (see [BGH], [BC3G]).