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Some Frequently Used Notation xiv

CHAPTER IV. INTRODUCTION TO ITÔ CALCULUS

TERMINOLOGY AND CONVENTIONS

- R*-processes and *L*-processes
- Usual conditions, etc.
- Important convention about time 0

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