Diffusions, Markov Processes, and Martingales

Volume 2: Itô Calculus

2nd Edition

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Preface

(*a*) *Welcome back on board!* You will have noticed that for this second leg of your journey, there are two pilots rather than one. D.W. is sure that you will be as delighted as he is that control is being shared with L.C.G.R.—amongst so many other things, just the man for a Wiley excursion!

We apologize for the considerable delay in departure. Anyone who knows what has been happening to British universities will need no further explanation, and will share our sadness.

(*b*) The book is meant to help the research student reach the stage where he or she can begin both to think up and tackle new problems and to read the up-to-date literature across a wide spectrum; and to persuade him or her that it is worth the effort.

We can say that we ourselves find the subject sufficiently good fun to have enjoyed the task of writing. (We even had some amusement from typing the manuscript ourselves with the very basic non-mathematical word-processor VIEW on the BBC micro. Occasionally, we got into trouble when trying to use global editing to substitute the most commonly occurring phrases for shorthand versions of our own devising. But, in the main, we were very satito's formulaied!)

(*c*) Chapter IV, *Introduction to Itô calculus*, is particularly concerned with developing the theory of the stochastic integral (of a previsible process) with respect to a continuous semimartingale, and with giving a large number of applications. Chung and (Ruth) Williams [1] would make a splendid companion volume for this chapter.

Chapter V, *Stochastic differential equations and diffusions*, presents first the theory of SDEs: existence and uniqueness for strong and weak solutions, martingale problems, etc. It has an extended treatment of 1-dimensional diffusions, and a huge attempt to introduce the very fashionable subject of stochastic differential geometry. Strongly recommended 'parallel' reading for this chapter: McKean's sparkling book [1] and the authoritative Ikeda and Watanabe [1].

Chapter VI, *The general theory*, presents *la théorie générale*: dual previsible projections, the Meyer decomposition theorem, the general integral, etc., with a chunky piece on excursions. The literature on the general theory is dominated by the masterly account by Dellacherie and Meyer ([1]) who created so much of it. Dellacherie's own very fine survey article [3], Jacod [2], Metivier and Pellaumail [1] should also be consulted.

In everything, the Russian literature, as represented by such important volumes as Gikhman and Skorokhod [1] and Liptser and Shiryayev [1], has its own characteristic style and special value.

(*d*) The book has a large bibliography, but this represents a small and rather haphazard selection of what we should have included. We apologize for the enormous number of very important papers which are omitted.

Numerous important topics are omitted too, or given treatment far too brief for their true significance. (Reviewers who find the previous sentence handy are free to use it without acknowledgement.) So, here are some guidelines on what you might move on to when your reading of our book is done.

(*e*) (i) *Large deviations.* The recent appearance of books by Stroock [4] and the grandmaster himself, Varadhan [1], would have made any efforts from us look silly. This is the only reason for our omission of this topic and for the (otherwise scandalous) omission from the bibliography of the historic papers by Donsker and Varadhan and by Ventcel and Freidlin.

(ii) *Malliavin calculus.* See § V.36.

(iii) *Large deviations and Malliavin calculus.* See Bismut [4], and also Elworthy and Truman [1, 2] for important work which provided motivation. Keep a look out for forthcoming work by Léandre.

(iv) *Markov processes.* The value of the classics mentioned in Volume 1—Blumenthal and Getoor [1], Getoor [1], and Meyer [3]—remains as great as ever. Sharpe [1] is sure to be a definitive account, as (of course) is that provided by later volumes of Dellacherie and Meyer [1]. (Volume 4 of the latter has arrived just as we are posting off the final proofs. Splendid to look forward to reading it!). The volumes in the 'Seminars on stochastic processes' (Çinlar, Chung and Getoor [1]) are important state-of-the-art reports.

Ethier and Kurtz [1] is a valuable source for much theory, for the establishment of weak-convergence results, etc. Liggett's account [1] of one of the most important application areas, interacting particle systems, is magisterial.

For a profound study of the relationship between Markov processes and semimartingales, see Çinlar, Jacod, Protter and Sharpe [1].

For applications to potential theory and complex analysis, see Doob [3], Durrett [1], and Port and Stone [3]. Two papers by Lyons [1, 2] are very much recommended.

(v) *Quantum theory.* So much has been achieved in interrelating quantum theory and probability that one hardly knows where to begin, but an excellent lead-in is provided by de Witte-Morette and Elworthy [1].

It is essential to realize that some of the finest work on probability is being done by people who are first and foremost mathematical physicists or functional analysts. See Simon [1, 2], Davies and Simon [1], Aizenmann and Simon [1], and the literature you can trace through them.

Local time and self-intersection local time have come to play a big part in the construction of quantum fields. See Geman and Horowitz [1], Rosen [1], Geman, Horowitz and Rosen [1], Le Gall [2, 3], Yor [4, 5] and then Dynkin [5, 6, 7] to begin your study in this area.

Whatever the philosophical problems, Nelson's *stochastic mechanics* is certainly prompting very interesting mathematics. See Nelson [1, 2] and Carlen [1, 2].

A fascinating theory of *non-commutative stochastic integrals* and of non-commutative SDEs has been created by Hudson and Parthasarathy. Meyer [1, 2] is a splendid attempt to make probabilists informed and involved.

(vi) *Measure-valued diffusions, random media, etc.*. Durrett [2] and Dawson and Gärtner [1] can be your 'open sesame' to what is sure to be one of the richest of Aladdin's caves.

(vii) *The Séminaires.* It is impossible to overstate our indebtedness to the famous *Séminaires de Probabilités*, originated by Meyer and developed by him (with help from Dellacherie and Weil) into an absolutely indispensable handbook, and now maintained as such in Azéma and Yor's expert hands. *Séminaire* XX: *Springer Lecture Notes in Mathematics Volume* 1204, contains an index to the series so far.

(*f*) *Further acknowledgements.* The work on this book has been done at the Universities of Wales (Swansea), Warwick and Cambridge, all of which deserve our thanks.

Most was done at Swansea where both of us spent very happy times. Special thanks to Aubrey Truman, Peter Townsend and Betty Williams.

We thank our colleagues at Cambridge for their warm welcome; and are pleased to acknowledge the help and advice we have received from many, especially Frank Adams, Keith Carne, David Kendall and James Norris.

Our best thanks to Sheila Williams, amanuensis extraordinary, who is just about to rediscover after a long period that there are such things as a dining-room table and a sideboard in the Williams household.

And, of course, our thanks to Charlotte Farmer, Robert Hambrook and the other staff of Wiley for making sure that it has become a reality; and to copy editors, and to wonderfully accurate typesetters.

Cambridge, October 1986 Chris Rogers
David Williams

Added, April 2000
Our thanks too to the staff of C.U.P., especially David Tranah, and also to the wonderfully accurate typesetters for their superb 'invisible mending'.

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CHAPTER V. STOCHASTIC DIFFERENTIAL EQUATIONS AND DIFFUSIONS

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Some Frequently Used Notation

$\mathscr{P}$,	the previsible σ-algebra; § IV.6.
b $\mathscr{P}$,	the space of bounded previsible processes; § IV.6
b $\mathscr{E}$,	the space of bounded elementary processes; § IV.6.
FV_0,	the space of (adapted) finite-variation processes null at zero; § IV.7.
IV_0,	the space of (adapted) integrable-variation processes null at zero; § IV.7.
IV $\mathscr{M}_0$,	the space of integrable-variation martingales null at zero; § IV.8.
lb $\mathscr{P}$,	the space of locally bounded previsible processes; § IV.10.
$\mathscr{M}_{0,\mathrm{loc}}$,	the space of local martingales null at zero; § IV.11.
FV $\mathscr{M}_{0,\mathrm{loc}}$,	the space of finite-variation local martingales null at zero; § IV.11.
UI $\mathscr{M}_0$,	the space of uniformly-integrable martingales null at zero; § IV.11.
$\mathscr{S}$,	the space of semimartingales; § IV.15.
$\mathscr{M}_0^2$,	the space of L^2-bounded martingales null at zero; § IV.24.
c $\mathscr{M}_0^2$,	the space of continuous L^2-bounded martingales null at zero; § IV.24.
d $\mathscr{M}_0^2$,	the space of purely discontinuous L^2-bounded martingales null at zero; § IV.24.
m $\mathscr{G}$,	($\mathscr{G}$ a σ-algebra), the space of $\mathscr{G}$-measurable functions.
b$\mathscr{G}$,	the space of bounded $\mathscr{G}$-measurable functions.
$[M]$, $[X]$,	quadratic-variation processes; §§ IV.26, 30, VI.36–38.
$\langle M \rangle$,	§ VI.34.
${}^{\circ}X$,	the optional projection of X; § VI.7
A^{p},	the dual previsible projection of A; §§ VI.1, 21, 23.
$\equiv$,	(which) is defined to equal.
$\uparrow\uparrow$,	$S_n \uparrow\uparrow t$ means: $S_n \leqslant S_{n+1} < t$ and $S_n \to t$.

$\mathbb{N} \equiv \{1, 2, 3, \ldots\}$, $\mathbb{Z}^+ \equiv \{0, 1, 2, \ldots\}$, $\mathbb{R}^+ \equiv [0, \infty)$, $\mathbb{R}^{++} \equiv (0, \infty)$.