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The Mandelbrot set is universal

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Abstract

We show small Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps.

1 Introduction

Fix an integer $d \geq 2$, and let $p_c(z) = z^d + c$. The *generalized Mandelbrot set* $M_d \subset \mathbb{C}$ is defined as the set of c such that the Julia set $J(p_c)$ is connected. Equivalently, $c \in M_d$ iff $p_c^n(0)$ does not tend to infinity as $n \to \infty$. The traditional Mandelbrot set is the quadratic version M_2.

A *holomorphic family of rational maps over* X is a holomorphic map

$$f : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

where X is a complex manifold and $\widehat{\mathbb{C}}$ is the Riemann sphere. For each $t \in X$ the family f specializes to a rational map $f_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, denoted $f_t(z)$. For convenience we will require that X is *connected* and that $\deg(f_t) \geq 2$ for all t.

The *bifurcation locus* $B(f) \subset X$ is defined equivalently as the set of t such that:

1. The number of attracting cycles of f_t is not locally constant;
2. The period of the attracting cycles of f_t is locally unbounded; or
3. The Julia set $J(f_t)$ does not move continuously (in the Hausdorff topology) over any neighborhood of t.

It is known that $B(f)$ is a closed, nowhere dense subset of X; its complement $X - B(f)$ is also called the *J-stable set* [MSS], [Mc2, §4.1].

As a prime example, $p_c(z) = z^d + c$ is a holomorphic family parameterized by $c \in \mathbb{C}$, and its bifurcation locus is ∂M_d. See Figure 1.

In this paper we show that *every* bifurcation set contains a copy of the boundary of the Mandelbrot set or its degree d generalization. The Mandelbrot sets M_d are thus *universal*; they are initial objects in the category of bifurcations, providing a lower bound on the complexity of $B(f)$ for all families f_t.

For simplicity we first treat the case $X = \Delta = \{t \ : \ |t| < 1\}$.

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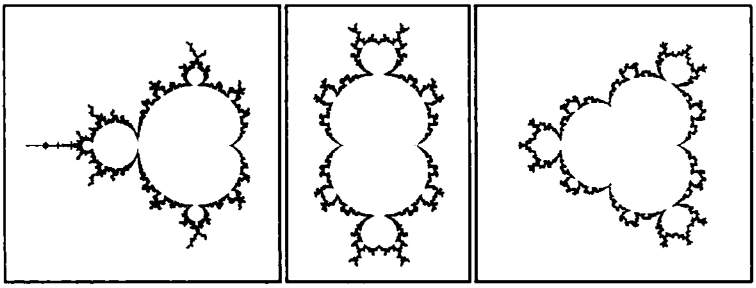

Figure 1. Mandelbrot sets of degrees 2, 3 and 4.

Theorem 1.1 *For any holomorphic family of rational maps over the unit disk, the bifurcation locus $B(f) \subset \Delta$ is either empty or contains the quasiconformal image of ∂M_d for some d.*

The proof (§4) shows that $B(f)$ contains copies of ∂M_d with arbitrarily small quasiconformal distortion, and controls the degrees d that arise. For example we can always find a copy of ∂M_d with $d \leq 2^{2\deg(f_t)-2}$, and generically $B(f)$ contains a copy of ∂M_2 (see Corollary 4.4). Since the Theorem is local we have:

Corollary 1.2 *Small Mandelbrot sets are dense in $B(f)$.*

There is also a statement in the dynamical plane:

Theorem 1.3 *Let f be a holomorphic family of rational maps with bifurcations. Then there is a $d \geq 2$ such that for any $c \in M_d$ and $m > 0$, the family contains a polynomial-like map $f_t^n : U \to V$ hybrid conjugate to $z^d + c$ with* $\mathrm{mod}(U - V) > m$.

Corollary 1.4 *If f has bifurcations then for any $\epsilon > 0$ there exists a t such that $f_t(z)$ has a superattracting basin which is a $(1+\epsilon)$-quasidisk.*

Proof. The family contains a polynomial-like map $f_t^n : U \to V$ hybrid conjugate to $p_0(z) = z^d$, a map whose superattracting basin is a round disk. Since $\mathrm{mod}(V - U)$ can be made arbitrarily large, the conjugacy can be made nearly conformal, and thus f_t has a superattracting basin which is a $(1+\epsilon)$-quasidisk. ∎

For applications to Hausdorff dimension we recall:

Theorem 1.5 (Shishikura) *For any $d \geq 2$, the Hausdorff dimension of ∂M_d is two. Moreover* $\mathrm{H.dim}(J(p_c)) = 2$ *for a dense G_δ of $c \in \partial M_d$.*

This result is stated for $d = 2$ in [Shi2] and [Shi1] but the argument generalizes to $d \geq 2$. Quasiconformal maps preserve sets of full dimension [GV], so from Theorems 1.1 and 1.3 we obtain:

Corollary 1.6 *For any family of rational maps f over Δ, the bifurcation set $B(f)$ is empty or has Hausdorff dimension two.*

Corollary 1.7 *If f has bifurcations, then* $\mathrm{H.dim}(J(f_t)) = 2$ *for a dense set of $t \in B(f)$.*[1]

For higher-dimensional families one has (§5):

Corollary 1.8 *For any holomorphic family of rational maps over a complex manifold X, either $B(f) = \emptyset$ or* $\mathrm{H.dim}(B(f)) = \mathrm{H.dim}(X) = 2\dim_{\mathbb{C}} X$.

Similar results on Hausdorff dimension were obtained by Tan Lei, under a technical hypothesis on the family f [Tan].

A family of rational maps f is *algebraic* if its parameter space X is a quasi-projective variety (such as $\mathbb{C}^n$) and the coefficients of $f_t(z)$ are rational functions of t. For example, $p_c(z) = z^d + c$ is an algebraic family over $X = \mathbb{C}$. Such families almost always contain bifurcations [Mc1]:

Theorem 1.9 *For any algebraic family of rational maps, either*

1. *The family is trivial (f_t and f_s are conformally conjugate for all $t, s \in X$); or*

2. *The family is affine (every f_t is critically finite and double covered by a torus endomorphism); or*

3. *The family has bifurcations ($B(f) \neq \emptyset$).*

Corollary 1.10 *With rare exceptions, any algebraic family of rational maps exhibits small Mandelbrot sets in its parameter space.*

[1] This set of t can be improved to a dense G_δ using Shishikura's idea of hyperbolic dimension.

This Corollary was our original motivation for proving Theorem 1.1.

As another application, for $t \in \mathbb{C}^{d-1}$ let

$$f_t(z) = z^d + t_1 z^{d-2} + \cdots + t_{d-1}$$

and let

$$\mathcal{C}_d = \{t \ : \ J(f_t) \text{ is connected}\}$$

denote the *connectedness locus*. Then we have:

Corollary 1.11 (Tan Lei) *The boundary of the connectedness locus has full dimension; that is,* $\mathrm{H.dim}(\partial\mathcal{C}_d) = \mathrm{H.dim}(\mathcal{C}_d) = 2d - 2$.

Proof. Consider the algebraic family $g_a(z) = z^d + az^{d-1}$, which for $a \neq 0$ has all but one critical point fixed under g_a. By Theorem 1.9, this family has bifurcations at some $a \in \mathbb{C}$. Then there is a neighborhood U of $(a, 0, \dots, 0) \in \mathbb{C}^{d-2}$ such that for $t \in U$ all critical points of f_t save one lie in an attracting or superattracting basin. If $t \in B(f) \cap U$, then the remaining critical point has a bounded forward orbit under f_t, but under a small perturbation tends to infinity. It follows that $B(f) \cap U = \partial\mathcal{C}_d \cap U \neq \emptyset$, and thus $\dim(\partial\mathcal{C}_d) \geq \dim B(f) = 2d - 2$. ∎

Remark. Rees has shown that the bifurcation locus has positive measure in the space of all rational maps of degree d [Rees]; it would be interesting to known general conditions on a family f such that $B(f)$ has positive measure in the parameter space X.

Acknowledgements. I would like to thank Tan Lei for sharing her results which prompted the writing of this note. Special cases of Theorem 1.1 were developed independently by Douady and Hubbard [DH, pp.332-336] and Eckmann and Epstein [EE].

2 Families of rational maps

In this section we begin a more formal study of maps with bifurcations.

Definitions. A *local bifurcation* is a holomorphic family of rational maps $f_t(z)$ over the unit disk Δ, such that $0 \in B(f)$.

The following natural operations can be performed on f to construct new local bifurcations:

1. *Coordinate change:* replace f_t by $m_t \circ f_t \circ m_t^{-1}$, where $m : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic family of Möbius transformations.

2. *Iteration:* replace $f_t(z)$ by $f_t^n(z)$ for a fixed $n \geq 1$.

3. *Base change:* replace $f_t(z)$ by $f_{\phi(t)}(z)$, where $\phi : \Delta \to \Delta$ is a nonconstant holomorphic map with $\phi(0) \in B(f)$.

The first two operations leave the bifurcation locus unchanged, while the last transforms $B(f)$ to $\phi^{-1}(B(f))$.

Marked critical points. We will also consider pairs (f, c) consisting of a local bifurcation and a *marked critical point*; this means $c : \Delta \to \widehat{\mathbb{C}}$ is holomorphic and $f_t'(c_t) = 0$. The operations above also apply to (f, c); a coordinate change replaces c_t with $m_t(c_t)$ and a base change replaces c_t with $c_{\phi(t)}$.

Misiurewicz points. A marked critical point c of f is *active* if its forward orbit

$$\langle f_t^n(c_t) \ : \ n = 1, 2, 3, \ldots \rangle$$

fails to form a normal family of functions of t on any neighborhood of $t = 0$ in Δ. A parameter t is a *Misiurewicz point* for (f, c) if the forward orbit of c_t under f_t lands on a repelling periodic cycle. If $t = 0$ is a Misiurewicz point, then either c is active or c_t is preperiodic for all t.

Proposition 2.1 *If c is an active critical point, then (f, c) has a sequence of distinct Misiurewicz points $t_n \to 0$.*

Proof. This is a traditional normal families argument. Choose any 3 distinct repelling periodic points $\{a_0, b_0, c_0\}$ for f_0, and let $\{a_t, b_t, c_t\}$ be holomorphic functions parameterizing the corresponding periodic points of f_t for t near zero. Since $\langle f_t^n(c_t) \rangle$ is not a normal family, it cannot avoid these three points, and any parameter t where $f_t^n(c_t)$ meets a_t, b_t or c_t is a Misiurewicz point. ∎

Ramification. Next we discuss the existence of univalent inverse branches for a single rational map $F(z)$. Let $d = \deg(F, z)$ denote the local degree of F at $z \in \widehat{\mathbb{C}}$; we have $d > 1$ iff z is a critical point of multiplicity $(d - 1)$. We say y is an *unramified preimage* of z if for some $n \geq 0$, $F^n(y) = z$ and $\deg(F^n, y) = 1$. We say z is *unramified* if it has infinitely many unramified preimages. In this case its unramified preimages accumulate on the full Julia set $J(F)$.

Proposition 2.2 *If z has 5 distinct unramified preimages then it has infinitely many.*

Proof. Let E be the set of all unramified preimages of z, and let C be the critical points of F. Then $F^{-1}(E) \subset E \cup C$, so if $|E|$ is finite then

$$d|E| = \sum_{z \in F^{-1}(E)} 1 + \text{mult}(f', z) \leq |F^{-1}(E)| + 2d - 2 \leq |E| + 4d - 4$$

and therefore $|E| \leq 4$. ∎

Corollary 2.3 *Let (f, c) be a local bifurcation with marked critical point. Then the set of t such that c_t is ramified for f_t is either discrete or the whole disk.*

Proof. By the previous Proposition, the ramified parameters are defined by a finite number of analytic equations in t. ∎

Proposition 2.4 *After a suitable base change, any local bifurcation f can be provided with an active marked critical point c such that c_0 is unramified for f_0.*

Remark. It is possible that all the active critical points are ramified at $t = 0$. The base change in the Proposition will generally not preserve the central fiber f_0.

Proof. The set $C = \{(t, z) \in \Delta \times \widehat{\mathbb{C}} \ : \ f_t'(z) = 0\}$ is an analytic variety with a proper finite projection to Δ. By Puiseux series, after a base change of the form $\phi(t) = \epsilon t^n$ all the critical points of f can be marked by holomorphic functions $\{c_t^1, \ldots, c_t^m\}$. Since $t = 0$ is in the bifurcation set, by [Mc2, Thm. 4.2], there is an i such that $\langle f_t^n(c_t^i)\rangle$ is not a normal family at $t = 0$. That is, c^i is an active critical point.

Next we show c^i can be chosen so that for generic t it is disjoint from the forward orbits of all other critical points. If not, there is a c^j and $n \geq 1$ such that $f_t^n(c_t^j) = c_t^i$ for all t. Then c^j is also active and we may replace c^i with c^j. If the replacement process were to cycle, then c^i would be a periodic critical point, which is impossible because it is active. Thus we eventually achieve a c^i which is generically disjoint from the forward orbits of the other critical points.

In particular, there is a t such that c_t^i is unramified for f_t. By Corollary 2.3, the set $R \subset \Delta$ of parameters where c_t^i is ramified is discrete. By Proposition 2.1, there are Misiurewicz points t_n for (f, c^i) with $t_n \to 0$. Choose n such that $t_n \notin R$, and make a base change moving t_n to zero; then c^i is active, and c_0^i is unramified for f_0. ∎

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Misiurewicz bifurcations. Let (f, c) be a local bifurcation with a marked critical point. We say (f, c) is a *Misiurewicz bifurcation* of degree d if

M1. $f_0(c_0)$ is a repelling fixed-point of f_0;

M2. c_0 is unramified for f_0;

M3. $f_t(c_t)$ is not a fixed-point of f_t, for some t; and

M4. $\deg(f_t, c_t) = d$ for all t sufficiently small.

Proposition 2.5 *For any local bifurcation (f, c) with c active and c_0 unramified, there is a base change and an $n > 0$ such that (f^n, c) is a Misiurewicz bifurcation.*

Remark. The delicate point is condition (M4). The danger is that for every Misiurewicz parameter t, the forward orbit of c_t might accidentally collide with another critical point before reaching the periodic cycle. We must avoid these collisions to make the degree of f_t^n at c_t locally constant.

Proof. There are Misiurewicz points $t_n \to 0$ for (f, c), and c_t is unramified for all t near 0, so after a base change and replacing f with f^n we can arrange that (f, c) satisfies conditions (M1), (M2) and (M3).

We can also arrange that $\deg(f_t, c_t) = d$ for all $t \neq 0$. However (M4) may fail because $\deg(f_t, c_t)$ may jump up at $t = 0$. This jump would occur if another critical of f_t coincides with c_t at $t = 0$.

To rule this out, we make a further perturbation of f_0. Let a_t locally parameterize the repelling fixed-point of f_t such that $f_0(c_0) = a_0$. Choose a neighborhood U of a_0 such that for t small, f_t is linearizable on U and U is disjoint from the critical points of f_t. (This is possible since $f_0'(a_0) \neq 0$.)

Let $b_t \in U - \{a_t\}$ be a parameterized repelling periodic point close to a_t. Then b_t has preimages in U accumulating on a_t. Choose s near 0 such that $f_s(c_s)$ hits one of these preimages (such an s exists by the argument principle and (M3)). For this special parameter, c_s first maps close to a_s, then remains in U until it finally lands on b_s. Since there are no critical points in U, we have $\deg(f_s^i, c) = d$ for all $i > 0$.

Making a base change moving s to $t = 0$, we find that (f^n, c) satisfies (M1-M4) for n a suitable multiple of the period of b_s. ∎

3 The Misiurewicz cascade

In this section we show that when a Misiurewicz point bifurcates, it produces a cascade of polynomial-like maps.

Definitions. A *polynomial-like map* $g : U \to V$ is a proper, holomorphic map between simply-connected regions with $\overline{U}$ compact and $\overline{U} \subset V \subset \mathbb{C}$ [DH]. Its *filled Julia set* is defined by

$$K(g) = \bigcap_{1}^{\infty} g^{-n}(V);$$

it is the set of points that never escape from U under forward iteration.

Any polynomial such as $p_c(z) = z^d + c$ can be restricted to a polynomial-like map $p_c : U \to V$ of degree d with the same filled Julia set. Moreover small analytic perturbations of $p_c : U \to V$ are also polynomial-like.

A degree d Misiurewicz bifurcation (f, c) gives rise to polynomial-like maps $f_t^n : B_0 \to B_n$, by the following mechanism. For small t, a small ball B_0 about the critical point c_t maps to a small ball B_1 close to, but not containing, the fixed-point of f_t. The iterates $B_i = f_t^i(B)$ then remain near the fixed-point for a long time, ultimately expanding by a large factor. Finally for suitable t, as B_i escapes from the fixed-point it maps back over B_0, resulting in a degree d map $f_t^n : B_0 \to B_n \supset B_0$. Since most of the images $\langle B_i \rangle$ lie in the region where f_t behaves linearly, the first-return map $f_t^n : B_0 \to B_n$ behaves like a polynomial of degree d.

This scenario leads to a cascade of families of polynomial-like maps, indexed by the return time n. Here is a precise statement.

Theorem 3.1 *Let (f, c) be a degree d Misiurewicz bifurcation, and fix $R > 0$. Then for all $n \gg 0$, there is a coordinate change depending on n such that $c_t = 0$ and*

$$f_t^n(z) = z^d + \xi + O(\epsilon_n)$$

whenever $|z|, |\xi| \leq R$. Here $t = t_n(1 + \gamma_n \xi)$, t_n and γ_n are nonzero, and γ_n, t_n and ϵ_n tend to zero as $n \to \infty$.

The constants in $O(\cdot)$ above depend on f and R but not on n.

The proof yields more explicit information. Let $\lambda_0 = f_0'(f_0(c_0))$ be the multiplier of the fixed-point on which c_0 lands, and let r be the multiplicity of intersection of the graph of c_t and the graph of this fixed-point at $t = 0$. Then for $t = t_n$, the critical point c_t is periodic with period n, and we have:

$$t_n \sim C\lambda_0^{-n/r}, \tag{3.1}$$
$$\gamma_n = C'\lambda_0^{-n/(d-1)}, \quad \text{and} \tag{3.2}$$
$$\epsilon_n = n(|\lambda_0|^{-n/r} + |\lambda_0|^{-n/(d-1)}), \tag{3.3}$$

for certain constants C, C' depending on f. Due to the choice of roots, there are r possibilities for t_n and $(d-1)$ for γ_n; the Theorem is valid for all choices.

Finally for ξ fixed and $t = t_n(1+\gamma_n\xi)$, the map f_t^n is polynomial-like near c_t for all $n \gg 0$, and in the *original* z-coordinate its filled Julia set satisfies

$$\operatorname{diam} K(f_t^n) \asymp |\lambda_0|^{-n/(d-1)}.$$

Notation. We adopt the usual conventions: $a_n = O(b_n)$, $a_n \asymp b_n$, $a_n \sim b_n$ and $n \gg 0$ mean $|a_n| < C|b_n|$, $(1/C)|b_n| < |a_n| < C|b_n|$, $a_n/b_n \to 1$ and $n \geq N$, where C and N are implicit constants.

Proof. We will make several constructions that work on a small neighborhood of $t = 0$. First, let a_t parameterize the repelling fixed-point of f_t such that $a_0 = f_0(c_0)$. Let $\lambda_t = f_t'(a_t)$ be its multiplier. There is a holomorphically varying coordinate chart $u = \phi_t(z)$ defined near $z = a_t$ such that

$$\phi_t \circ f_t \circ \phi_t^{-1}(u) = \lambda_t u \tag{3.4}$$

for u near 0. We call $u = \phi_t(z)$ the *linearizing coordinate*; note that $u = 0$ at a_t.

We next arrange that $u = 1$ is an unramified preimage of c_t. Since c_0 is unramified by (M2), its unramified preimages accumulate on a_0. Let b_0 be one such preimage, with $f_0^p(b_0) = c_0$ and b_0 in the domain of ϕ_0. Then b_0 prolongs to a holomorphic function b_t with $f_t^p(b_t) = c_t$. Replacing ϕ_t by $\phi_t(z)/\phi_t(b_t)$, we can assume $u = \phi_t(b_t) = 1$.

For small t, the composition $f_t^p \circ \phi_t^{-1}$ is univalent near $u = 1$. By applying a coordinate change $z \mapsto m_t(z)$, where m_t is a Möbius transformation depending on t, we can arrange that $c_t = 0$ and that

$$f_t^p \circ \phi_t^{-1}(u) = (u-1) + O((u-1)^2) \tag{3.5}$$

on $B(1,\epsilon)$.

Since $\deg(f_t, 0) = d$ for t near 0 by (M4), we have

$$\begin{aligned} \phi_t \circ f_t(z) &= \sum A_i(t) z^i && (3.6)\\ &= A_0(t) + A_d(0) z^d (1 + O(|z| + |t|)) && (3.7) \end{aligned}$$

with $A_d(0) \neq 0$. Here $A_0(t) = f_t(0)$ is the u-coordinate of the critical value. By (M3), c_t is not pre-fixed for all t, so there is an $r > 0$ such that

$$A_0(t) = t^r B(t) \tag{3.8}$$

where $B(0) \neq 0$.

Next for $n \gg 0$ we choose t_n such that

$$f_t^{1+n+p}(c_t) = c_t \quad \text{when } t = t_n. \tag{3.9}$$

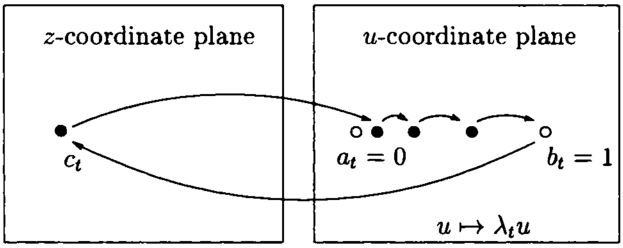

Figure 2. Visiting the repelling fixed-point

More precisely, for $t = t_n$ we will arrange that c_t maps first close to a_t, then lands after n iterates on b_t, and thus returns in p further iterates to c_t; see Figure 2. In the u-coordinate system, f_t is linear and $b_t = 1$, so the equation $f_t^{n+1}(c_t) = b_t$ becomes

$$\lambda_t^n A_0(t) = 1 \quad \text{when } t = t_n. \tag{3.10}$$

By the argument principle, for $n \gg 0$ this equation has a solution t_n close to any root of the approximation $\lambda_0^n t^r B(0) = 1$ obtained from (3.8). Moreover

$$t_n \sim B(0)^{-1} \lambda_0^{-n/r}$$

(verifying (3.1)), and t_n satisfies (3.9) because $f_t^p(b_t) = c_t$. (There are actually be r solutions for t_n for a given n; any one of the r solutions will do.)

We now turn to the estimate of $f_t^{1+n+p}(z)$ for (t, z) near $(t_n, 0)$. We will assume throughout that $t = t_n + s$ and that:

$$|z| \textit{ and } |s/t_n| \textit{ are } O(\Lambda^{-n/(d-1)}) \tag{3.11}$$

where $\Lambda = |\lambda_0| > 1$. (To see this is the correct scale at which to work, suppose $\operatorname{diam}(B) \asymp \operatorname{diam} f_t^{1+n+p}(B)$, where B is a ball centered at $z = 0$. Then $\operatorname{diam} f_t(B) \asymp (\operatorname{diam} B)^d$, and f_t^n is expanding by a factor of about Λ^n, while f_t^p is univalent, so we get $\operatorname{diam} B \asymp \Lambda^n(\operatorname{diam} B)^d$, or $\operatorname{diam} B \asymp \Lambda^{-n/(d-1)}$. Similarly $|f_t^{1+n+p}(0)| \asymp \Lambda^n(s/t_n)t_n^r \asymp (s/t_n) = O(\operatorname{diam} B)$ when s is as above.)

It is also convenient to set

$$\widetilde{\Lambda} = \min(\Lambda^{1/(d-1)}, \Lambda^{1/r}) > 1,$$

so that we may assert:

$$z \textit{ and } t \textit{ are } O(\widetilde{\Lambda}^{-n}). \tag{3.12}$$