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J. Eells
Cambridge/Warwick

B. Fuglede
Copenhagen

With a preface by M. Gromov

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CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by
Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521773119

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First published 2001

A catalogue record for this publication is available from the British Library

Library of Congress cataloguing in publication data
Eells, James, 1926-
Harmonic maps between Riemannian polyhedra / J. Eells & B. Fuglede; with a preface
by M. Gromov
p. cm. - (Cambridge tracts in mathematics)
Includes bibliographical references.
ISBN 0 521 77311 3
1. Harmonic maps. 2. Riemannian manifolds. I. Fuglede, Bent. II. Title. III. Series.
QA614.73.E353 2001
514´.74-dc21 00-054670

ISBN 978-0-521-77311-9 Hardback

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Gromov's Preface

Harmony and Harmonicity

If you fall in love with harmonic functions your mathematician's soul will never come to rest unless you comprehend the origin of their irresistible appeal and beauty. And if you are bent on spaces, manifolds and maps you start researching for the geometric habitat of harmonicity.

In 1964 Eells and Sampson found the promised land not limited to functions but encompassing harmonic maps between (almost) arbitrary Riemannian manifolds. Yet one's greed for generality has not been quenched and the urge to extend harmonicity to rugged terrains of singular spaces could not be contained for long.

Here the story of this begins. There are two players in the harmonic mappings game: the source space X and the target Y. Suppose we are granted a *harmonic structure* on X, that is a distinguished space (or rather a sheaf) of $\mathbb{R}$-valued functions on X regarded as "harmonic". Then one can, under suitable assumptions, define another space (sheaf) consisting of corresponding "subharmonic functions" on X. Similarly, one needs distinguished functions on Y: these should be thought of as "convex functions". A map $f : X \to Y$ is declared "harmonic" if the pull-back of every "convex function" on Y is "subharmonic" on X.

Now a hard choice is to be made: how much of "Riemannian" is one willing to sacrifice for the sake of generality (singularity) and harmonicity? The authors decide in favour of "measurable Riemannian" for both X and Y, where X is a (rather general) topological polyhedron and Y is allowed an arbitrary local topology modified by the negative sign restriction on the curvature. In other words, the dimensions of X and Y are limited, for most part, to be finite, fractality (e.g. subellipticity) is not admitted and foliated structures are not allowed. By paying this price one arrives at a full fledged harmonic theory on X, extending Nash–De Giorgi–Moser, which then perfectly welds with the negative curvature on Y. It is some three hundred pages of smooth ride.

Misha Gromov
May 15, 2000

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Authors' Preface

During the past thirty-five years harmonic maps between smooth Riemannian manifolds have played a significant role in geometry and elliptic analysis. They provide an especially rich mixture of classical potential theory and the Riemannian geometry of maps. Relevant guidelines are described in [EL 1978, 1988], mostly without proof, but with full references.

About eight years ago it became apparent that the notion of harmonicity should be expanded to include maps between certain singular spaces. In particular, (a) for applications to rigidity Gromov (see [GS 1992]) has shown that Riemannian polyhedra, and more generally the geodesic spaces of Alexandrov and Busemann (Definition 2.7), are natural targets; (b) certain Riemannian polyhedra (e.g., normal complex analytic spaces) are natural domains for harmonic maps.

This monograph is a research essay on harmonic maps between admissible Riemannian polyhedra (definitions in Chapter 4), these being prototypes of the relevant singular spaces. While harmonic spaces (in the sense of Brelot) are natural domains for harmonic functions, we have not been able to study harmonic maps in that generality, not even when adding a suitable Dirichlet space structure to obtain a notion of energy. We have discovered, however, that admissible Riemannian polyhedra are both geodesic, harmonic and Dirichlet spaces, more precisely hypoelliptic Dirichlet spaces in the sense of Feyel and de La Pradelle [FP 1978]. These polyhedra illustrate clearly our main ideas, and provide a wealth of examples as well. Thus harmonic maps (especially when presented in their variational context, via the Dirichlet integral) between Riemannian polyhedra are our main object of study.

A particular novelty in our presentation is the use of the fine topology of H. Cartan (the weakest topology in which all subharmonic functions are continuous), and its intimate relation to quasitopological concepts (defined in terms of capacity). This leads among other things to the quasicontinuity of finite energy maps in the sense of Korevaar and Schoen [KS 1993] into geodesic spaces, also in our setting of maps with polyhedral domain.

In spite of their importance, we do not treat applications in detail, for

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that simply would take us too far afield. However, here are three such – the first having smooth Riemannian domains and singular targets, the second having singular domains and smooth Riemannian targets:

- p-adic super-rigitity for lattices of rank one in the isometry groups of the quaternionic hyperbolic space and the Cayley plane, [GS 1992] and [Co 1992].
- Representation of integral homology classes of a compact Riemannian manifold by harmonic maps of compact oriented normal circuits [E 1997]. The proof is based on the method in [EF 1991] – except for the case of 2-dimensional homology classes, which can be so represented, using [Cha 1988].
- Reduction methods, as in [ER 1993, Chapter IV]. The idea (Examples 12.2 and 13.5) is to obtain an equivariant harmonic map $\psi : M \to N$ between smooth Riemannian manifolds, starting from a harmonic map $\varphi : M/K \to N/L$ between orbit spaces, where K (resp., L) is a compact group of isometries of M (resp., N); and ψ covers φ.

We have discussed various aspects of our text with many colleagues and with much profit; in particular, S. Hildebrandt, J. Jost, N. Korevaar, B. Lackey, L. Lemaire, C. Plaut, M. Ramachandran, R. Schoen, Richard Stong, K.–T. Sturm, D. Toledo and M. Wolf. Hereby we record our special thanks to all!

It is our pleasure to record our appreciation of the effort of M. Gromov, who graciously responded to our invitation to write a preface. And to T. Serbinowski for providing Chapter 15, a version of an unpublished part of his Thesis.

We further thank the referees, whose constructive comments have substantially improved the final version.

The second named author is grateful for all the facilities at the Department of Mathematics, University of Copenhagen, made available to him in his retirement. In this connection we are indebted to Anders Thorup and Lene Kørner, who helped us with the AmSTeX file.

Cambridge/Warwick — James Eells
Copenhagen — Bent Fuglede
September 2000

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