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Introduction

This chapter consists of two parts: (1) A brief summary of the theory of smooth harmonic maps between Riemannian manifolds; that should provide background comparison to the main theme of the monograph. (2) A description of Riemannian polyhedra as harmonic and geodesic spaces; and energy-minimizing maps between them.

The following terminology for maps $f : X \to Y$ between metric spaces is adopted throughout this monograph (cf. Chapter 4):

- f is said to be *Lip continuous* if f is *locally* Lipschitz;
- f is called a *Lip homeomorphism* if f is bijective and if f and f^{-1} are both Lip continuous. Equivalently, f is a *locally* bi-Lipschitz bijection.

Similarly, f is said to be *Hölder continuous* if f satifies a *local* Hölder condition.

If X is a manifold or polyhedron with boundary, that boundary is denoted bX. The topological boundary of a subset A of a topological space is denoted ∂A.

The smooth framework

Let M be a smooth manifold without boundary, endowed with a smooth positive definite Riemannian metric g. The pair (M, g) is called a *Riemannian manifold.* We shall only consider *connected* Riemannian manifolds. Each tangent vector space $T_x(M)$ is Euclidean, with inner product $\langle \cdot, \cdot \rangle_x = g(x)(\cdot, \cdot)$. The volume measure μ_g on (M, g) has local representation of the form $\sqrt{\det g(x)}\, dx$. Associated to g is a metric d_M on M compatible with its underlying topology – defined before Definition 2.7 in terms of the Riemannian length of paths in (M, g).

For simplicity of exposition, assume here that M is compact. Define the Hilbert space $W^{1,2}(M)$ (a Sobolev space) as the completion of the space of

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smooth functions $u : M \to \mathbb{R}$ with respect to the inner product

$$\int_M \{u(x)v(x) + \langle \nabla u(x), \nabla v(x)\rangle_x\}\, d\mu_g(x),$$

where ∇u denotes the gradient field of u, and

$$\langle \nabla u(x), \nabla v(x)\rangle_x = g^{ij}(x)\frac{\partial u(x)}{\partial x^i}\frac{\partial v(x)}{\partial x^j}.$$

The *Dirichlet integral*, or *energy functional*, of u is

$$E(u) = \frac{1}{2}\int_M |\nabla u(x)|_x^2\, d\mu_g(x).$$

Its *Euler–Lagrange operator* is the Laplace–Beltrami operator Δ, characterized by

$$\frac{dE(u_t)}{dt}\Big|_{t=0} = -\int \Delta u_t(x)\frac{du_t(x)}{dt}\Big|_{t=0} d\mu_g(x).$$

Here (u_t) is a 1-parameter deformation of $u = u_0$. Locally,

$$\Delta u = (\det g)^{-1/2}\partial_i(g^{ij}(\det g)^{1/2}\partial_j u),$$

where $\partial_i = \partial/\partial x^i$, $i = 1, \cdots, n$.

A function u on an open set $U \subset M$ is *harmonic* if $\Delta u = 0$ in the distributional sense. *Such a function is smooth* (after correction on a null set). For our main theme, two fundamental properties are emphasized:

(a) *Dirichlet problem.* For any small closed ball B in (M, g) and smooth function $f : \partial B \to \mathbb{R}$ on its boundary, there is a unique smooth function $u : B \to \mathbb{R}$ such that $u_{|\partial B} = f$ and $u_{|\,\mathrm{int}\, B}$ is harmonic.

(b) *Harnack's monotone convergence property.* For any increasing sequence (u_n) of harmonic functions on a connected open subset U of M, the pointwise limit function $u = \sup_n u_n$ is either harmonic on U, or $u \equiv \infty$.

Most of the qualitative aspects of smooth linear potential theory (as in [GT 1998, Part I], [Hel 1969]) can be derived from those two properties. That is the essence of Brelot's harmonic spaces, see Chapter 2.

Now take a second smooth Riemannian manifold (N, h), and consider a continuous map $\varphi : M \to N$ of class $W^{1,2}(M, N)$, i.e., the components φ^k in terms of local coordinates y^k, $k = 1, \cdots, n$, are of class $W^{1,2}$. The *energy density* of φ is the integrable function $e(\varphi) : M \to \mathbb{R}$ given almost everywhere by

$$e(\varphi)(x) = \tfrac{1}{2}\,\mathrm{trace}_{g(x)}(\varphi^* h) = \tfrac{1}{2} g^{ij}(x) \frac{\partial \varphi^\alpha(x)}{\partial x^i} \frac{\partial \varphi^\beta(x)}{\partial x^j} h_{\alpha\beta}(\varphi(x)).$$

In other words, $2e(\varphi)(x)$ is the square of the Hilbert–Schmidt norm of the differential $d\varphi(x)$, viewed as a linear map $T_x(M) \to T_{\varphi(x)}(N)$ between the indicated Euclidean tangent spaces. The *energy* of φ is defined as the integral

$$E(\varphi) = \int_M e(\varphi) d\mu_g.$$

If φ is smooth, the Euler–Lagrange operator of that energy functional E evaluated on the map φ is the *tension field* $\tau(\varphi)$, which is a vector field along the map φ:

$$\frac{dE(\varphi_t)}{dt}\Big|_{t=0} = -\int_M \Big\langle \tau(\varphi)(x), \frac{d\varphi_t(x)}{dt}\Big|_{t=0} \Big\rangle_x \, d\mu_g(x).$$

Again, (φ_t) is a 1-parameter deformation of $\varphi = \varphi_0$.

A continuous map $\varphi \in W^{1,2}(M, N)$ is said to be *weakly harmonic* if it satisfies $\tau(\varphi) \equiv 0$, or in local coordinates

$$\Delta \varphi^k = -(\Gamma^k_{\alpha\beta} \circ \varphi) \langle \nabla \varphi^\alpha, \nabla \varphi^\beta \rangle, \tag{1.1}$$

in the weak sense. It follows that φ (after correction on a null set) is C^∞-smooth [ES 1964, §8]. Here is a simple bootstrap argument for this: The right hand member of (1.1) is integrable (over the pre-image of the coordinate patch in question), and it follows from (1.1) that φ is of class $C^{1,\lambda}$ for some $\lambda > 0$. The right hand member of (1.1) is therefore of class C^λ, and so, by (1.1), φ is $C^{2,\lambda}$, etc.

There are weakly harmonic maps which are not continuous, provided that $\dim M > 2$ [EL 1978, §3.5].

A continuous map $\varphi \in W^{1,2}(M, N)$ is weakly harmonic iff φ is *harmonic*, i.e., φ is *bi-locally E-minimizing* in the sense described below in the section Harmonic maps (even for maps from an admissible Riemannian polyhedron, cf. Definition 12.1).

Continuous maps which minimize E in a given homotopy class of maps $M \to N$ are clearly bi-locally E-minimizing and hence harmonic.

A continuous map $\varphi : M \to N$ between smooth Riemannian manifolds is said to be *totally geodesic* if it maps geodesics of M linearly to geodesics of

N. Equivalently, if φ preserves connections [EL 1983, Proposition 2.21]. *A totally geodesic map is harmonic.*

Note that if $N = \mathbb{R}$ then the harmonic maps $M \to \mathbb{R}$ are just the harmonic functions. More generally, a special case of harmonic map $\varphi : M \to N$ arises when φ is *horizontally weakly conformal*; i.e., if for any point $x \in M$ at which the differential $d\varphi(x) \neq 0$, its restriction to the orthogonal complement of $\ker(d\varphi(x))$ in $T_x(M)$ is conformal and surjective. *A map $\varphi : M \to N$ is harmonic and horizontally weakly conformal iff it is a harmonic morphism*; i.e., for every function v which is harmonic on an open subset V of N, the composition $v \circ \varphi$ is a harmonic function on $\varphi^{-1}(V) \subset M$.

T. Ishihara has characterized harmonic and totally geodesic maps in a similar spirit: Say that a smooth function $v : V \to \mathbb{R}$ is *convex* if its Hessian $\nabla^2 v$ is positive semidefinite. And is *subharmonic* if Δv $(= \operatorname{trace} \nabla^2 v) \geqslant 0$. Then: *A continuous map $\varphi : M \to N$ is harmonic, resp. totally geodesic, iff it pulls germs of convex functions on N back to germs of subharmonic, resp. convex, functions on M* [Ish 1979].

The remaining sections of this Introduction describe the broad lines of development in our text – first for Riemannian polyhedra as harmonic and geodesic spaces; and then for maps, especially for various E-minimizers.

Harmonic and Dirichlet spaces

There are several natural frameworks for the qualitative aspects of potential theory on locally compact spaces. For instance (Chapter 2), [Br 1957; 1958a,b; 1959]:

1. *Brelot harmonic spaces*, identified and developed [Br 1969] as spaces characterized (as already indicated) through

   (a) local unique solvability of the Dirichlet problem for continuous boundary functions, and
   (b) Harnack's monotone convergence property.

A more general notion, for somewhat different purposes, has been studied by Bauer [Bau 1966]. See [CC 1972], [Bau 1984] for comparisons.

2. *Dirichlet spaces of Beurling and Deny* [BD 1959], [De 1970], [FOT 1994]. Here the starting point is a Dirichlet form E, abstracting key properties of the classical Dirichlet integral D (in particular, Beurling's observation that D decreases under normal contractions).

By a theorem of Feyel and de La Pradelle [FP 1978], certain so-called hypoelliptic Dirichlet spaces determine Brelot harmonic spaces.

It has been established in important work by Biroli and Mosco [BM 1991; 1995] that for certain so-called admissible Dirichlet spaces, the extremals of the Dirichlet form E are Hölder continuous, in the presence of a suitable Poincaré inequality. Admissibility requires that

(a) a certain Carathéodory distance d on the underlying space X be a pseudometric whose topology is the given one, and
(b) the Radon measure of the Dirichlet space satisfy a ball doubling condition.

Sturm [St 1995b] has shown (even without (b)) that (X, d) (if complete) *is a geodesic space* (meaning briefly that any two points in the same component can be joined by a geodesic, i.e., a rectifiable path whose length is the distance between the points).

The traditional examples of Brelot and Dirichlet spaces are analytical in character: Spaces of solutions of second order linear elliptic operators, possibly with discontinuous coefficients ([Her 1964; 1965], [HH 1969; 1972], [Bo 1967], [La 1980]).

Riemannian polyhedra

By way of contrast, this monograph is primarily concerned with harmonic spaces derived from and motivated by geometric considerations. In more detail: A connected locally finite n-dimensional simplicial polyhedron X is called *admissible* (cf. [Chen 1995]) if

(i) X is dimensionally n-homogeneous, i.e., every simplex is a face of an n-simplex, and
(ii) X is locally $(n-1)$-chainable, i.e., relative to some triangulation, for any simplex s, any two n-simplexes containing s are joinable by a chain of contiguous $(n-1)$- and n-simplexes containing s.

Examples 8.2 and 8.3 indicate that both these conditions are necessary for our purposes.

A polyhedron X becomes a *Riemannian polyhedron* when endowed with a Riemannian structure g, defined by giving on each maximal simplex s of X a Riemannian metric g_s equivalent to a Euclidean metric on s. The components of g_s need not be smooth, but merely *bounded measurable.*

Adapting constructions by De Cecco and Palmieri [CP 1988; 1990], an *intrinsic distance* d_X (Carathéodory distance) is defined on X, which thereby becomes a *length space*, and hence a geodesic space, if complete (Proposition 4.1).

On an admissible Riemannian polyhedron X the Sobolev space $W^{1,2}(X)$, defined as the completion of a suitable space of Lipschitz continuous functions in the Sobolev norm, *is a Dirichlet space* (Proposition 5.1).

Harmonic functions on X

On an admissible Riemannian polyhedron X define the *energy functional* $E : W^{1,2}_{\mathrm{loc}}(X) \to \mathbb{R}$ by

$$E(u) = \frac{1}{2}\int_X |\nabla u|^2 d\mu_g.$$

Weakly harmonic (and subharmonic) functions are defined using the energy functional, and characterized variationally (Proposition 5.2, Theorem 5.2).

Theorem 5.3 is a maximum principle for weakly subharmonic functions of class $W^{1,2}(X)$, of the sort due to Hervé [Her 1964] in the case where X is a domain in $\mathbb{R}^n$ carrying a Riemannian metric with bounded measurable coefficients; our proof is different.

Harnack's inequality for weakly harmonic functions on an admissible Riemannian polyhedron X is established (Theorem 6.1), involving a careful and leisurely adaptation of the argument of Moser [Mos 1961]. In the spirit of De Giorgi [Gi 1957] and Moser [Mos 1960], it is shown in Theorem 6.2 that *every weakly harmonic function on X is Hölder continuous.* (Simple examples illustrate that such functions may not be Lipschitz continuous.) Furthermore (Theorem 6.3), *every locally uniformly bounded family of harmonic functions on X is locally uniformly Hölder equicontinuous.*

Continuous, weakly harmonic functions are called *harmonic.* Altogether we are led to Theorem 7.1 which states that *an admissible Riemannian polyhedron has a natural Brelot harmonic space structure; and its harmonic funtions are Hölder continuous* (i.e., have Hölder continuous versions). It was first proved by Hervé [Her 1964] in the case where X is a domain in $\mathbb{R}^n$ with a bounded measurable Riemannian structure.

Now suppose that X satisfies the following Poincaré inequality (always fulfilled locally):

$$\left(\int_K |u|\, d\mu_g\right)^2 \leqslant c(K)E(u)$$

for any $u \in \mathrm{Lip}_c(X)$ (the compact Lip functions $u : X \to \mathbb{R}$). Then $\sqrt{E(u)}$ is a *norm* on $\mathrm{Lip}_c(X)$, and *the completion $L_0^{1,2}(X)$ of $\mathrm{Lip}_c(X)$ in this norm is a regular Dirichlet space of diffusion type* (Proposition 7.3). In view of the continuity of weakly harmonic functions the theorem of Feyel and de La Pradelle [FP 1978] applies to the Dirichlet space $L_0^{1,2}(X)$, and produces the Brelot harmonic space of Theorem 7.1. That and constructions of Hervé [Her 1962] combine to ensure (always in the presence of the above Poincaré inequality) that *every admissible Riemannian polyhedron has a unique sym-*

metric Green function G, Hölder continuous off the diagonal (Theorem 7.3, Propositon 7.4).

We show that $G(x, y)$ is bounded above and below by constant multiples of $d_X(x, y)^{2-n}$ when x and y vary in a compact set and $n > 2$ (Theorem 7.4). Our proof of these estimates exploits the full local polyhedral structure of an admissible Riemannian polyhedron, thus permitting us to adapt a method used by Littman, Stampacchia and Weinberger [LSW 1963] in their proof of Lemma 7.2 and Theorem 7.4 for the particular case of a ball X in $\mathbb{R}^n$ endowed with a Riemannian metric with bounded measurable components. A key ingredient in their proof is the use of a suitable homothetic transformation in order to reduce the case of a variable point y to that of a fixed y. When X is an admissible Riemannian polyhedron, a finite number of such homotheties are applied in our proof of Theorem 7.4. The same technique is further used in the proof of the Poincaré inequality for functions (Theorem 5.1), and in a Harnack inequality for positive functions harmonic off a point (Proposition 6.2).

In their paper [BM 1995] Biroli and Mosco studied potential theory on a very general type of metric space endowed with a Dirichlet form of diffusion type, and even allowing for degeneracy. As an alternative approach to the Harnack inequality and the Hölder continuity of weakly harmonic functions on an admissible Riemannian polyhedron, one could use our preparatory results to verify the remaining axioms in the Biroli–Mosco theory in the present case of an admissible Riemannian polyhedron (X, g). This leads to the existence of the Green function on small balls in (X, g) and to our estimates of it, invoking a uniform estimate of the volume of balls with variable centre in a compact subset of X (Lemma 4.4).

Geometric examples

Various admissible Riemannian polyhedra are presented in Chapter 8. Included are the following examples which are also *circuits* (i.e., satisfying (i) and a global version of (ii) above, and also requiring that every $(n-1)$-simplex is a face of exactly one or two n-simplexes, see Chapter 4):

- Smooth Riemannian manifolds, with or without boundary. (The associated harmonic functions are the local solutions u of the Laplace–Beltrami equation $\Delta u = 0$ in the interior, and having vanishing normal derivative at the boundary, see Remark 5.3.) Also triangulable Riemannian Lipschitz manifolds.
- Riemannian joins of smooth Riemannian manifolds.
- Conical singular Riemannian spaces (Cheeger [Chee 1980; 1983]).

- Normal complex analytic spaces (Giesecke [Gie 1964] and Łojasiewicz [Ło 1964]).

In a somewhat different context (Examples 8.12 and 8.13): Let K be a compact group of isometries of a complete smooth Riemannian manifold M, and $\pi : M \to M/K$ the orbit projection. The Brelot harmonic sheaf $\mathcal{H}_M$ of M determines the direct image sheaf $\pi_* \mathcal{H}_M$ on M/K, a Brelot harmonic sheaf there.

Analogously, a Riemannian orbifold has such a harmonic space structure. For instance, the leaf space $M/\mathcal{F}$ of a Riemannian foliation $\mathcal{F}$ with closed leaves of a Riemannian manifold M (Reinhart [Rei 1961]).

Maps between polyhedra

Suppose that (X, g) is an admissible Riemannian polyhedron of dimension m (compact, for simplicity) and that g is simplexwise smooth. Let (Y, d_Y) be any separable metric space (to begin with).

A *Riemannian domain* is understood to be a connected open subset of a Riemannian manifold (M, g) whose metric completion is compact in M.

For suitable maps φ of a Riemannian domain into Y an energy density $e(\varphi)$ was defined by Korevaar and Schoen [KS 1993], via a deep subpartitioning lemma, as a certain limit of approximate energy densities $e_\varepsilon(\varphi)$; the energy $E(\varphi)$ then equals $\int e(\varphi)\, d\mu_g$. See Chapter 14 for a summary of that construction.

Building on this concept of energy, and its properties, we consider analogously the following approximate energy density of a measurable map $\varphi : (X, g) \to (Y, d_Y)$:

$$e_\varepsilon(\varphi)(x) = \int_{B_X(x,\varepsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\varepsilon^{m+2}} d\mu_g(x').$$

The *energy* $E(\varphi)$ of φ is defined as

$$E(\varphi) = \sup_{f \in C_c(X,[0,1])} \Big(\limsup_{\varepsilon \to 0} \int_X f\, e_\varepsilon(\varphi)\, d\mu_g \Big) \quad (\leqslant \infty).$$

It is shown that a map $\varphi : X \to Y$ has finite energy iff there is an integrable function $e(\varphi)$ on X (called the *energy density* of φ) such that $e_\varepsilon(\varphi) \to e(\varphi)$ as $\varepsilon \to 0$, in the sense of weak convergence of measures. This is also equivalent to φ *having a quasicontinuous version with restrictions of finite energy in the sense of Korevaar–Schoen to the top-dimensional simplexes s of X*. The sum of these energies $E(\varphi_{|s})$ then equals $E(\varphi)$ (Theorem 9.1). (A map $\varphi : X \to Y$ is said to be quasicontinuous if φ has continuous restrictions to closed sets with complements of arbitrarily small capacity.)

In case Y is a Riemannian C^1-manifold (N, h) without boundary, we also have the elementary concept of energy of a quasicontinuous map $\varphi : X \to N$, the energy density being given by $e(\varphi) = \mathrm{trace}_g\, \varphi^* h$, in coordinate patches V on N:

$$e(\varphi) = (h_{\alpha\beta} \circ \varphi)\langle \nabla\varphi^\alpha, \nabla\varphi^\beta \rangle \qquad \text{in } \varphi^{-1}(V).$$

This makes immediate sense, by covariance, if φ is continuous or if N is itself a patch. In any event, the pre-images $\varphi^{-1}(V)$ are *quasiopen*, and the components $\varphi^1, \cdots, \varphi^n$ are of class $W^{1,2}(U)$ in the sense of Kilpeläinen and Malý, [KilM 1992, §2] (cf. the next to last section of Chapter 7 below). The energy of φ is now defined by $E(\varphi) = \int_X e(\varphi)\, d\mu_g$.

This concept of energy of maps φ of (X, g) into a Riemannian manifold N is identified (Lemma 9.3) with the one suggested by Nash's isometric embedding theorem. The previous requirement that the Riemannian metric g of X be simplexwise smooth is not needed for this energy concept. With that requirement, however, *the two concepts of energy of maps into manifolds are identical*, provided e.g., that φ is continuous or that N is compact. In either of these cases, $e_\varepsilon(\varphi) \to e(\varphi)$ as $\varepsilon \to 0$ holds both in L^1-norm and pointwise almost everywhere in X, when φ has finite energy (Theorem 9.2).

There is a similar result for maps into a Riemannian polyhedron Y with continuous Riemannian metric h (Theorem 9.3). When (Y, h) admits a Riemannian embedding in some $\mathbb{R}^q$ we recover the elementary expression for energy used in [GS 1992] and [Chen 1995].

The energy functional E on $W^{1,2}(X, Y)$ is lower semicontinuous, and $W^{1,2}(X, Y)$ has a Rellich-style precompactness property if Y is also compact.

Assuming that Y admits a bi-Lipschitz embedding in a Euclidean space (this holds for any compact Riemannian polyhedron), Poincaré's inequality for maps $X \to Y$ of finite energy is derived from that for functions (Proposition 9.1).

This allows us to establish the Hölder continuity of locally E-minimizing maps into suitable targets, by adapting a line of arguments of Jost [J 1997a], with underlying work by Jäger–Kaul [JK 1979], Caffarelli [Caf 1982], Giaquinta–Giusti [GG 1982], Giaquinta–Hildebrandt [GH 1982], Meier [Mei 1984] and Sturm [St 1995a].

With our first concept of energy for maps from an admissible Riemannian polyhedron X with simplexwise smooth Riemannian metric g we obtain the following local regularity property, assuming that Y is a simply connected complete Riemannian polyhedron of nonpositive curvature (in the sense of A.D. Alexandrov): *If $\varphi : X \to Y$ is a local E-minimizer, then φ is Hölder continuous* (Theorem 10.1).

Likewise, assuming that (Y, d_Y) is a complete Riemannian polyhedron with $\operatorname{curv}(Y, d_Y) \leqslant K$ for some $K > 0$: Let $\varphi : X \to Y$ be a local E-minimizer, and suppose that the range of φ is contained in a compact convex set $V \subset Y$ of diameter $< \pi/(2\sqrt{K}\,)$ such that geodesics in V are uniquely determined by their endpoints. Then φ is Hölder continuous (Theorem 10.2).

In particular, every continuous locally E-minimizing map φ of X into a Riemannian polyhedron Y with upper bounded curvature is Hölder continuous (Corollary 10.3).

Similar results with the second energy concept and hence with a manifold target N are obtained in Propositions 12.1, 12.2 and Corollary 12.2. Some limitation on the size of the range of φ is necessary, even for maps between manifolds, see [HKW 1977, §6] (cf. Example 12.3 below).

With the above regularity theorems at hand it is possible to extend, by known direct methods of variational theory, classical results on existence of harmonic maps between Riemannian manifolds to the present setting of maps between Riemannian polyhedra.

Consider first the case of a target Y of nonpositive curvature. A key fact is that this curvature restriction implies strong convexity properties of E [GS 1992, §4], [KS 1993, Chapter 2] and [J 1994, §2].

For the case of *free homotopy* we obtain, assuming that X and Y are compact Riemannian polyhedra, X being admissible and Y of nonpositive curvature: *Every homotopy class of continuous maps $X \to Y$ has an E-minimizer, and any such is Hölder continuous* (Theorem 11.1).

A more detailed analysis gives the following uniqueness property: *If $\varphi_0, \varphi_1 : X \to Y$ are homotopic E-minimizers which agree at some point of X, then φ_0 coincides with φ_1 on X* (Corollary 11.1).

Theorem 11.1 is due to

(a) Eells and Sampson [ES 1964] in case X and Y are both smooth Riemannian manifolds; the E-minimizers are smooth.

(b) Gromov and Schoen [GS 1992] when X is a smooth Riemannian manifold. The E-minimizers are Lipschitz continuous.

Korevaar and Schoen [KS 1993] have extended [GS 1992], dropping the polyhedral restriction on Y, permitting Y to be any geodesic space (compact and of nonpositive curvature). Then, as mentioned above, E is defined as a limit of approximating energy integrals.

(c) Chiang [Chi 1990], in case X is a Riemannian orbifold and Y a smooth Riemannian manifold.