Topics in Finite and Discrete Mathematics

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Contents

1. Preliminaries

1.1 Sets

A *set* is a collection of elements. If the set A consists of the n elements $a_1, a_2, \ldots, a_n$ then we express this by writing

$$A = \{a_1, a_2, \ldots, a_n\}.$$

Thus, for instance, the set consisting of all the integers between 6 and 10 is given by

$$B = \{6, 7, 8, 9, 10\}.$$

A set can be defined either by specifying all its elements, as just shown, or by specifying a defining property for its elements. Thus, the set B could have been defined as

$$B = \{\text{integers } i : |8 - i| \leq 2\}.$$

That is, B could have been defined as the set of all integers i such that the distance between i and 8 is less than or equal to 2.

A set consisting of a finite number of elements is said to be a *finite set*, whereas one consisting of an infinite number of elements is said to be an *infinite set*. The set $\mathcal{N}$ of all the nonnegative integers is an example of an infinite set. It is convenient to define the set that does not consist of any elements; we call this the *null* set and denote it by $\emptyset$.

We use the notation $a \in A$ to indicate that a is an element of A, and we use the notation $a \notin A$ to indicate that a is not a member of A.

Example 1.1a Let S be the set of all possible outcomes when a pair of dice are rolled. By an "outcome" we mean the pair (i, j), where i is the number of the side on which the first die lands and j is the number of the side for the second die. Then, the set of all outcomes that result in the sum of the dice being equal to 7 can be expressed as

$$S_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

or, alternatively, as

$$S_7 = \{(i, j) \in S : i + j = 7\}. \qquad \square$$

If every element of A is also an element of B, then we say that A is a *subset* of B and write $A \subset B$. By this definition, every set is a subset of itself and hence, for example, $A \subset A$. Also, since there are no elements in the null set, it follows that every element of $\emptyset$ is also an element of A; thus, $\emptyset$ is a subset of every other set. If $A \subset B$ and $B \subset A$ then we may write $A = B$. That is, the sets A and B are said to be equal if every element of A is in B and every element of B is in A.

If A and B are sets then we define the new set $A \cup B$, called the *union* of A and B, to consist of all elements that are in A or in B (or in both). Also, we define the *intersection* of A and B, written either as $A \cap B$ or just AB, to consist of all elements that are in both A and B.

Example 1.1b In Example 1.1a, if A is the set of all outcomes for which the sum of the dice is 5 and if B is the set of outcomes for which the value of the second die exceeds that of the first die by the amount 3, then we have

$$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\} \quad \text{and} \quad B = \{(1, 4), (2, 5), (3, 6)\};$$

also,

$$A \cup B = \{(1, 4), (2, 3), (3, 2), (4, 1), (2, 5), (3, 6)\} \quad \text{and}$$
$$AB = \{(1, 4)\}.$$

If we define C to be the set of all outcomes whose sum is equal to 6, then $AC = \emptyset$ because there are no outcomes whose sum is both 5 and 6. $\square$

A set is said to be a *universal set* if it contains (as subsets) all other sets under consideration. Let $\mathcal{U}$ be a universal set. For any set A, the set A^c, called the *complement* of A, is defined to be the set containing all the elements of the universal set $\mathcal{U}$ that are *not* in A.

Venn diagrams are often used to graphically represent sets. The universal set $\mathcal{U}$ is represented as consisting of all the points in a large rectangle, and sets are represented as consisting of all the points in circles within the rectangle. Particular sets of interest are indicated by shading

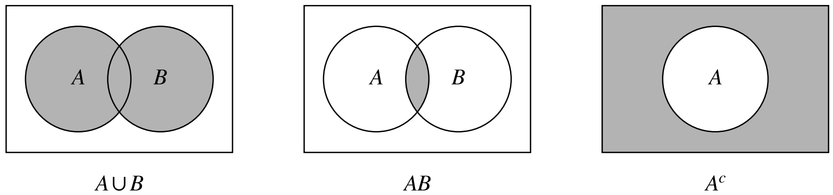


Figure 1.1

appropriate regions of the diagram. For instance, the Venn diagrams of Figure 1.1 indicate the sets $A \cup B$, AB, and A^c.

The operation of forming unions and intersections of sets obey certain rules that are similar to the rules of algebra. We list a few of them as follows:

Commutative laws: $A \cup B = B \cup A$, $AB = BA$.
Associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$, $(AB)C = A(BC)$.
Distributive laws: $(A \cup B)C = AC \cup BC$, $AB \cup C = (A \cup C)(B \cup C)$.

These relations are verified by showing that any element that is contained in the left-hand set is also contained in the right-hand one, and vice versa. For instance, to prove that

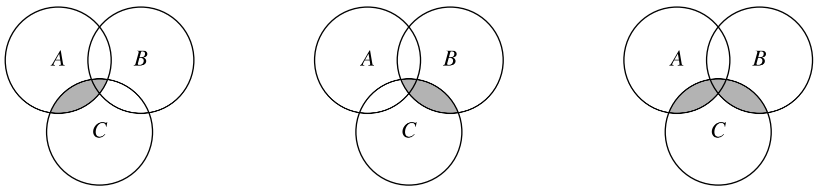
$$(A \cup B)C = AC \cup BC,$$

note that if $x \in (A \cup B)C$ then $x \in C$ and x is also in either A or B. If $x \in A$, then it is in AC and so is in $AC \cup BC$; similarly, if $x \in B$, then it is in BC and so is in $AC \cup BC$. Thus, $x \in AC \cup BC$, showing that

$$(A \cup B)C \subset AC \cup BC.$$

To go the other way, suppose that $y \in AC \cup BC$. Then y is either in both A and C or in both B and C. Therefore, we can conclude that y is in C and is in at least one of the sets A and B. But this means that $y \in (A \cup B)C$, showing that

$$AC \cup BC \subset (A \cup B)C,$$



AC BC $(A \cup B)C$

Figure 1.2

and the verification is complete. (The result could also be shown by using Venn diagrams; see Figure 1.2.)

We also define the intersection and union of more than two sets. Specifically, for sets $A_1, \ldots, A_n$ we define $\bigcup_{i=1}^{n} A_i$, the union of these sets, to consist of all elements that are in A_1, or in A_2, or in $A_3, \ldots,$ or in A_n; that is, $\bigcup_{i=1}^{n} A_i$ is the set of all elements that are in at least one of the sets A_i, $i = 1, \ldots, n$. Similarly, we define $\bigcap_{i=1}^{n} A_i$, the intersection of these sets, to consist of all elements that are in each of the sets A_i, $i = 1, \ldots, n$.

1.2 Summation

If we let s be the sum of the four numbers x_1, x_2, x_3, x_4 then we can write

$$s = x_1 + x_2 + x_3 + x_4.$$

More compactly, we can use the summation notation $\sum$. Using this latter notation, we write

$$s = \sum_{i=1}^{4} x_i,$$

which means that s is equal to the sum of the x_i values as i ranges from 1 to 4. More generally, for $j \leq n$, we use the notation

$$s = \sum_{i=j}^{n} x_i$$

to mean that

$$s = x_j + x_{j+1} + \cdots + x_n.$$

Example 1.2a If $x_i = i^2$, find $\sum_{i=3}^{6} x_i$.

***Solution*.**

$$\sum_{i=3}^{6} x_i = x_3 + x_4 + x_5 + x_6 = 9 + 16 + 25 + 36 = 86. \qquad \square$$

If S is a specified set of integers, then we use the notation

$$\sum_{i \in S} x_i$$

to represent the sum of all the values x_i that have indices in S.

Example 1.2b If $S = \{2, 4, 6\}$ then

$$\sum_{i \in S} x_i = x_2 + x_4 + x_6. \qquad \square$$

Consider the sum $T = \sum_{i=0}^{2} x_{2+i}$. Because T is equal to $x_2 + x_3 + x_4$, it follows that we can also express T as $T = \sum_{j=2}^{4} x_j$. Therefore, we see that

$$\sum_{i=0}^{2} x_{2+i} = \sum_{j=2}^{4} x_j.$$

When equating the right-hand summation to the left, we say that we are making the change of variable $j = 2 + i$. That is, summing the values x_{2+i} as i ranges from 0 to 2 is the same as summing the values x_j as j ranges from 2 to 4.

Example 1.2c Making the change of variable $j = n - i$ in the summation $\sum_{i=0}^{n} x_{n-i}$ gives the equivalent sum $\sum x_j$ as j ranges between n and 0. That is,

$$\sum_{i=0}^{n} x_{n-i} = \sum_{j=0}^{n} x_j. \qquad \square$$

We are sometimes interested in numbers that are expressed in the form $x_{i,j}$, where i and j both take values in some region. A quantity that is often of interest is the following "double sum" $D = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j}$, where

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_{i,j} \right).$$

Now arrange the numbers $x_{i,j}$ $(i = 1, \ldots, n,\ j = 1, \ldots, m)$ in the following row–column array, which has the number $x_{i,j}$ in row i, column j.

$$\begin{array}{ccccccc}
x_{1,1} & x_{1,2} & x_{1,3} & \ldots & x_{1,j} & \ldots & x_{1,m} \\
x_{2,1} & x_{2,2} & x_{2,3} & \ldots & x_{2,j} & \ldots & x_{2,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{i,1} & x_{i,2} & x_{i,3} & \ldots & x_{i,j} & \ldots & x_{i,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n,1} & x_{n,2} & x_{n,3} & \ldots & x_{n,j} & \ldots & x_{n,m}
\end{array}$$

Because $\sum_{j=1}^{m} x_{i,j}$ is just the sum of the m array elements in row i, it follows that the double sum D is equal to the sum of the row sums. In other words, D is equal to the sum of all the elements in the array. Since the sum of all the array values can also be obtained by adding all the column sums and since the sum of the values of column j is $\sum_{i=1}^{n} x_{i,j}$, we have the following result.

Proposition 1.2.1

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} = \sum_{j=1}^{m} \sum_{i=1}^{n} x_{i,j}.$$

A corollary of this proposition is the following useful result.

Corollary 1.2.1

$$\sum_{i=1}^{n} \sum_{j=1}^{i} x_{i,j} = \sum_{j=1}^{n} \sum_{i=j}^{n} x_{i,j}.$$

Proof. Consider data values $x_{i,j}$, where i and j both takes values from 1 to n and where $x_{i,j} = 0$ when $j > i$. Then apply Proposition 1.2.1. □

A pictorial proof of Corollary 1.2.1 is obtained by noting that its left-hand side,

$$\sum_{i=1}^{n}\sum_{j=1}^{i} x_{i,j} = \sum_{j=1}^{1} x_{1,j} + \sum_{j=1}^{2} x_{2,j} + \cdots + \sum_{j=1}^{n} x_{n,j},$$

is equal to the sum of all the row sums whereas the right-hand side,

$$\sum_{j=1}^{n}\sum_{i=j}^{n} x_{i,j} = \sum_{i=1}^{n} x_{i,1} + \sum_{i=2}^{n} x_{i,2} + \cdots + \sum_{i=n}^{n} x_{i,n},$$

is the sum of all the column sums in the following array.

$$\begin{array}{lllll} x_{1,1} & & & & \\ x_{2,1} & x_{2,2} & & & \\ x_{3,1} & x_{3,2} & x_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{array}$$

Example 1.2d

$$\begin{aligned} \sum_{i=1}^{3}\sum_{j=1}^{i}(i-j) &= \sum_{j=1}^{1}(1-j) + \sum_{j=1}^{2}(2-j) + \sum_{j=1}^{3}(3-j) \\ &= 0+1+3 = 4, \\ \sum_{j=1}^{3}\sum_{i=j}^{3}(i-j) &= \sum_{i=1}^{3}(i-1) + \sum_{i=2}^{3}(i-2) + \sum_{i=3}^{3}(i-3) \\ &= 3+1+0 = 4. \end{aligned}$$

□

Since $x\sum_{j=1}^{m} y_j = \sum_{j=1}^{m} xy_j$, it follows that

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{m} y_j\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_i\right)y_j = \sum_{j=1}^{m}\sum_{i=1}^{n} x_i y_j.$$

Example 1.2e Expand $(x_1 + \cdots + x_n)^2$.

Solution.

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j\right)$$
$$= \sum_i \sum_j x_i x_j$$
$$= \sum_i \left(x_i x_i + \sum_{j \neq i} x_i x_j\right)$$
$$= \sum_i x_i^2 + \sum_i \sum_{j \neq i} x_i x_j.$$

For instance, the preceding yields

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + x_1 x_2 + x_2 x_1 = x_1^2 + x_2^2 + 2x_1 x_2. \qquad \square$$

Similar to the notation for summations is our notation $\prod$ for products,

$$\prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_n.$$

Example 1.2f

$$\prod_{i=1}^{4} i = 1 \cdot 2 \cdot 3 \cdot 4 = 24. \qquad \square$$

1.3 Mathematical Induction

Suppose that we have an infinite collection of statements, denoted $S_1, S_2, \ldots$, and that we want to prove that they are all true. A proof by *mathematical induction* is obtained in the following manner:

(i) first prove that S_1 is true;
(ii) then show that, for any n, whenever S_n is true then S_{n+1} is also true.

Once (i) and (ii) are established, then from (i) we know that S_1 is true; which implies by (ii) that S_2 is true; which implies that S_3 is true; and so on. Thus, it follows that all of the S_n are true.

We now illustrate the use of mathematical induction by a series of examples.

Example 1.3a Prove that there are 2^n subsets of a set consisting of n elements.

Solution. In order to prove this by mathematical induction, we must first prove it for $n = 1$. But this is immediate, for if the set consists of a single element (i.e., if the set is $\{s\}$) then it has the two subsets $\emptyset$ and $\{s\}$, where $\emptyset$ is the empty set. Thus, part (i) of the mathematical induction approach is shown. To show part (ii), *assume* that the result is true for all sets of size n (this is called the *induction hypothesis*) and then consider a set S of size $n+1$. Focus attention on one of the elements of S, call it s, and let S' denote the set consisting of the n other elements of S. Because every subset of S that does not contain s is a subset of S', it follows from the induction hypothesis that there are 2^n subsets of S that do not contain s. On the other hand, since any subset of S that contains s can be obtained by adding s to a subset of S', it also follows from the induction hypothesis that there are 2^n of these subsets. Thus the total number of subsets of S is

$$2^n + 2^n = 2^n(1+1) = 2^{n+1},$$

and the result is proved. □

Example 1.3b For integer n, which is larger: 2^n or n^2?

Solution. Let us try a few cases:

$$\begin{aligned} 2^1 &= 2, & 1^2 &= 1; \\ 2^2 &= 4, & 2^2 &= 4; \\ 2^3 &= 8, & 3^2 &= 9; \\ 2^4 &= 16, & 4^2 &= 16; \\ 2^5 &= 32, & 5^2 &= 25; \\ 2^6 &= 64, & 6^2 &= 36. \end{aligned}$$

Thus, based on this enumeration, a reasonable conjecture is that $2^n > n^2$ for all values of $n \geq 5$. To prove this, we start by showing it to be true when $n = 5$; this was demonstrated by our preceding calculations. So now assume that, for some n $(n \geq 5)$,

$$2^n > n^2.$$

We must show that the preceding implies that $2^{n+1} > (n+1)^2$, which may be accomplished as follows.

First, note that

$$2^{n+1} = 2 \cdot 2^n > 2n^2,$$

where the inequality follows from the induction hypothesis. Hence, it will suffice to show that, for $n \geq 5$,

$$2n^2 \geq (n+1)^2$$

or (equivalently)

$$2n^2 \geq n^2 + 2n + 1$$

or

$$n^2 - 2n - 1 \geq 0$$

or

$$(n-1)^2 - 2 \geq 0$$

or

$$n - 1 \geq \sqrt{2},$$

which follows because $n \geq 5$. □

Example 1.3c Derive a simple expression for the following function:

$$f(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Solution. Again, let us begin by calculating the value of $f(n)$ for small values of n, hoping to discover a general pattern that we can then prove by mathematical induction. Such a calculation gives

$$f(1) = 1/2,$$
$$f(2) = 1/2 + 1/6 = 2/3,$$
$$f(3) = 2/3 + 1/12 = 3/4,$$
$$f(4) = 3/4 + 1/20 = 4/5.$$

Thus, a reasonable conjecture is that

$$f(n) = \frac{n}{n+1}.$$

Let us now prove this by induction. Since it is true when $n = 1$, assume that it is valid also for some other n and consider $f(n+1)$. We have

$$\begin{aligned}
f(n+1) &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \\
&= f(n) + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \quad \text{(by the induction hypothesis)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}.
\end{aligned}$$

Thus, the result is established. (As in any situation where one has proven a particulary nice result by mathematical induction, it pays to see if there is a more direct argument that establishes and also *explains* the result; see Exercise 1.18.) □

Example 1.3d If one has unlimited access to five-cent and seven-cent stamps, show that any postage value greater than or equal to 24 cents can be exactly met.

Solution. First note that a postage of 24 can be obtained by 2 fives and 2 sevens. Now assume that for some $n \geq 24$ the postage value n can be exactly hit with a combination of five- and seven-cent stamps, and suppose that we desire postage of value $n+1$. To obtain this exact amount, consider the combination that adds up to n. If it contains at least 2 seven-cent stamps, then trade 2 sevens for 3 fives to obtain the postage value $n+1$. If the combination adding to n contains at least 4 fives, replace them by 3 sevens to obtain the value $n+1$. Thus, the result is shown if the combination adding up to n contains either at least 2 sevens or at least 4 fives. The alternative is that it contains at most 1 seven and at most 3 fives; but this would imply that $n \leq 22$, which is not the case. Thus, the result is shown. □

Example 1.3e Show that, for any positive integer n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Solution. We need to show that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

This is true for $n = 1$, since both sides are equal to 1. So let us assume that it is true for some integer n. To verify it for $n+1$, we reason as follows:

$$\begin{aligned} 1 + 2 + \cdots + n + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \quad \text{(by the induction hypothesis)} \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

and the induction proof is complete. □

Example 1.3f Verify that, for any value $x \neq 1$ and positive integer n,

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$

Solution. Let us use induction. When $n = 1$, the identity says that

$$1 + x = \frac{1 - x^2}{1 - x},$$

which is true because

$$1 - x^2 = (1 - x)(1 + x).$$

So assume that the identity is true for a specified n. To prove that it remains true when n is increased by 1, note the following:

$$\begin{aligned}\sum_{i=0}^{n+1} x^i &= \sum_{i=0}^{n} x^i + x^{n+1} \\ &= \frac{1 - x^{n+1}}{1 - x} + x^{n+1} \quad \text{(by the induction hypothesis)} \\ &= \frac{1 - x^{n+1} + x^{n+1} - x^{n+2}}{1 - x} \\ &= \frac{1 - x^{n+2}}{1 - x}.\end{aligned}$$

Thus, the identity is also valid for $n + 1$, which shows that it is true for all n. □

Example 1.3g In a round-robin tennis tournament, every pair of competitors play a match. Show that if such a tournament were played with n players then there is a labeling of the players $p_1, p_2, \ldots, p_n$ such that

$$p_1 \text{ beat } p_2,\ p_2 \text{ beat } p_3,\ \ldots,\ p_{n-1} \text{ beat } p_n. \tag{1.1}$$

Solution. The verification is by induction. The result is immediate when $n = 2$, so suppose it to be true whenever there are n players and consider the case when there are $n + 1$. Put one of the players, call her p, aside. Then, by the induction hypothesis, there is an ordering of the other n players such that (1.1) holds. If p did not beat any of the other n players then

$$p_1 \text{ beat } p_2,\ p_2 \text{ beat } p_3,\ \ldots,\ p_{n-1} \text{ beat } p_n,\ p_n \text{ beat } p.$$

On the other hand, if p won at least one match then, with i equal to the smallest integer such that p beat p_i,

$$p_1 \text{ beat } p_2,\ \ldots,\ p_{i-1} \text{ beat } p,\ p \text{ beat } p_i,\ \ldots,\ p_{n-1} \text{ beat } p_n.$$

Thus the result is true whenever there are $n + 1$ players, which completes the induction proof. □

The following result, although intuitively obvious, is quite useful.

Proposition 1.3.1 *Every finite nonempty set of numbers A has a smallest and a largest element.*

Proof. We shall show by induction that A always has a smallest and a largest element whenever A is a set of n numbers. This is true when $n = 1$ (since the lone number in A is both the smallest and largest number of A), so assume it to be true for all sets of n numbers. Let A be a set consisting of $n + 1$ numbers, say $A = \{a_1, \ldots, a_n, a_{n+1}\}$. Then, by the induction hypothesis, the subset $\{a_1, \ldots, a_n\}$ has a smallest and largest element (say, a_i and a_j resp.). But then A has a smallest element, namely the smaller of a_i and a_{n+1}, and a largest element, namely the larger of a_j and a_{n+1}. This completes the induction and, since a finite nonempty set must contain n elements for some n, also establishes the result. □

The well-ordering property of the integers is a simple consequence of Proposition 1.3.1.

Corollary 1.3.1 (Well-Ordering Property of Positive Integers) *Every set A containing at least one positive integer has a smallest positive integer.*

Proof. Let n be a positive integer in A. Any integer in A that is larger than n cannot be the smallest positive integer in A. Hence it follows that, if the set $A_n = \{i : i \text{ is an integer}, i \in A, i \le n\}$ has a smallest member, then that integer is also the smallest positive integer in A. But since A_n is a finite set, it has a smallest member. □

We now use mathematical induction to prove a well-known mathematical result.

Proposition 1.3.2 (Hardy's Lemma) *Consider two collections of numbers,*

$$a_1 \le a_2 \le \cdots \le a_n \quad \textit{and} \quad b_1 \le b_2 \le \cdots \le b_n,$$

and suppose that we have to make n disjoint pairs from these collections, each pair consisting of one a and one b. Then the sum of the products of the members of each pair is maximized when a_i is paired with b_i for each $i = 1, \ldots, n$.

Proof. When $n = 2$ we must show that

$$a_1b_1 + a_2b_2 \ge a_1b_2 + a_2b_1,$$

which is equivalent to

$$a_2(b_2 - b_1) \ge a_1(b_2 - b_1)$$

or

$$(a_2 - a_1)(b_2 - b_1) \ge 0;$$

this is true beause both factors are nonnegative. So assume that the result is true whenever there are n numbers in each collection, and suppose now that there are $n + 1$ values $a_1 \le \cdots \le a_{n+1}$ and $n + 1$ values $b_1 \le \cdots \le b_{n+1}$ to be paired up. Consider any pairing of the $n + 1$ a and b values in which a_1 is not paired with b_1 – rather, a_1 is paired with (say) b_i. Then, aside from this individual pairing, there remain n members of each set to be paired up:

$$a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n+1}$$

to be paired up with

$$b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n+1}.$$

By the induction hypothesis, the pairing that maximizes the sum of the products from the remaining pairings – call this pairing M – will pair a_2 (the smallest a) with b_1 (the smallest b). Thus the best pairing that pairs up a_1 with b_i will also pair up a_2 with b_1. But by the result shown when $n = 2$, it is at least as good to pair up a_1 with b_1 and a_2 with b_i and then pair the others as does M. Thus, we need only consider pairings that pair up a_1 and b_1; by the induction hypothesis, the best one of this type also pairs up a_i with b_i for each $i = 2, \ldots, n + 1$, which completes the proof. □

The mathematical induction proof technique sometimes uses the following, "strong" version of induction.

Strong Version of Mathematical Induction To prove that all the statements $S_1, S_2, \ldots$ are true:

(i) prove that S_1 is true;
(ii) show that, for any n, if $S_1, \ldots, S_n$ are all true then S_{n+1} is also true.

The strong version is valid because – once (i) and (ii) are established – from (i) we know that S_1 is true; which implies by (ii) that S_2 is true; which implies, since S_1 and S_2 are both true, that S_3 is true; and so on. Indeed, the strong version proof that all of the statements S_n are true is equivalent to the standard mathematical induction proof of the statements S_n^* $(n \geq 1)$, where S_n^* is the statement that $S_1, \ldots, S_n$ are all true.

Example 1.3h Let $a_1 = 3$, $a_2 = 7$, and

$$a_n = 3a_{n-1} - 2a_{n-2}, \quad n = 3, 4, \ldots.$$

Find an explicit expression for a_n and prove your result.

Solution. Let us start by evaluating some of the early values of a_n in the hope of discovering a pattern. This yields

$$\begin{aligned} a_1 &= 3, \\ a_2 &= 7, \\ a_3 &= 21 - 6 = 15, \\ a_4 &= 45 - 14 = 31, \\ a_5 &= 93 - 30 = 63, \\ a_6 &= 189 - 62 = 127. \end{aligned}$$

It is not difficult to spot that $a_n = 2^{n+1} - 1$ for all of the values of n between 1 and 6. To prove that this holds for all n, assume that $a_k = 2^{k+1} - 1$ for all values less than or equal to n. Then

$$\begin{aligned} a_{n+1} &= 3a_n - 2a_{n-1} \\ &= 3(2^{n+1} - 1) - 2(2^n - 1) \\ &= 3 \cdot 2^{n+1} - 3 - 2^{n+1} + 2 \\ &= 2 \cdot 2^{n+1} - 1, \end{aligned}$$

which completes the induction proof since $2 \cdot 2^{n+1} = 2^{n+2}$. □

1.4 Functions

A real-valued function is a rule that associates a real number to every element x of a set X. The function is symbolically represented as f, and the value associated to the element x is designated as $f(x)$. The set X is called the *domain* of f.

Example 1.4a If X is the set of integers, then the function

$$f(i) = i^2$$

associates to each integer i the value i^2. □

Definition Let f be a function whose domain is the set of integers. We say that f is an *increasing* function if, for every integer i,

$$f(i + 1) \geq f(i).$$

Similarly, we say that f is a *decreasing* function if, for every integer i,

$$f(i + 1) \leq f(i).$$

Example 1.4b Are the following functions increasing, decreasing, or neither?

(a) $f(i) = 5i$.

(b) $f(i) = i^2$.

(c) $f(i) = \log(i)$.

(d) $f(i) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases}$

Solution. The function in (a) is increasing. The function in (b) is increasing if the domain of the function is the set of nonnonegative integers; however, if the domain is the set of all integers then it is neither increasing nor decreasing. Assuming that the domain of the function in (c) is the set of positive integers, then the function is increasing. The function in (d) is neither increasing nor decreasing. □

If f is an increasing function on the integers, then it can be shown that

$$f(i) \leq f(j) \quad \text{if } \ i < j. \tag{1.2}$$

One way to establish (1.2) is to note the sequence of inequalities

$$f(i) \leq f(i+1) \leq f(i+2) \leq \cdots \leq f(j).$$

A more formal proof would be to use mathematical induction to prove that, for all $n \geq 0$,

$$f(n+i) \geq f(i).$$

The preceding is true when $n = 1$; assuming it true for n yields

$$\begin{aligned} f(n+1+i) &\geq f(n+i) && \text{(by the definition of an increasing function)} \\ &\geq f(i) && \text{(by the induction hypothesis),} \end{aligned}$$

which completes the more formal induction proof of equation (1.2).

If f and g are functions defined on the same domain X, then we say that

$$f \leq g \quad \text{(equivalently, } g \geq f\text{)}$$

if, for all $x \in X$,

$$f(x) \leq g(x).$$

Similarly, we say that

$$f = g$$

if, for all $x \in X$,

$$f(x) = g(x).$$

The function that associates the same value c to every element in X is said to be a *constant* function and is denoted by c. Hence, the notation

$$f \leq c$$

means that $f(x) \leq c$ for all $x \in X$. A function f of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is said to be a *polynomial* function. The next example uses mathematical induction to verify a sufficient condition for a polynomial function to be positive whenever $x \geq 1$.

Example 1.4c Prove that

$$\sum_{i=0}^{n} a_i x^i > 0 \quad \text{for all } \; x \geq 1,$$

provided that

$$\begin{aligned} a_n &> 0, \\ a_{n-1} + a_n &> 0, \\ a_{n-2} + a_{n-1} + a_n &> 0, \\ &\vdots \\ a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n &> 0. \end{aligned}$$

Solution. Suppose the preceding conditions on $a_0, \ldots, a_n$ and assume that $x \geq 1$. Let

$$\begin{aligned} P(0) &= a_n, \\ P(1) &= a_{n-1} + x a_n = a_{n-1} + xP(0), \\ P(2) &= a_{n-2} + x a_{n-1} + x^2 a_n = a_{n-2} + xP(1), \\ &\vdots \\ P(j) &= a_{n-j} + x a_{n-j+1} + \cdots + x^j a_n = a_{n-j} + xP(j-1), \\ &\qquad j = 1, \ldots, n. \end{aligned}$$

Thus, the objective is to show that

$$P(n) > 0 \quad \text{if } \; x \geq 1.$$

We will accomplish this by using mathematical induction to prove that, for all $j = 0, \ldots, n$,

$$P(j) \geq a_{n-j} + a_{n-j+1} + \cdots + a_n. \tag{1.3}$$

Since the RHS of (1.3) is assumed to be positive, the result would then be proven. Equation (1.3) holds when $j = 0$, so assume that

$$P(j) \geq a_{n-j} + a_{n-j+1} + \cdots + a_n > 0.$$

Then,

$$\begin{aligned} P(j+1) &= a_{n-j-1} + xP(j) \\ &> a_{n-j-1} + P(j) \quad \text{(since } P(j) > 0 \text{ and } x \geq 1) \\ &\geq a_{n-j-1} + a_{n-j} + a_{n-j+1} + \cdots + a_n \\ &\qquad\qquad \text{(by the induction hypothesis),} \end{aligned}$$

and the proof by mathematical induction is complete. □

Functions on the same domain can be combined to form new functions. For instance, if f and g are functions on the integers then so are the functions $f + g$ and fg, defined by

$$f + g(i) = f(i) + g(i),$$

$$fg(i) = f(i)g(i).$$

That is, the values associated with i by the functions $f + g$ and fg are, respectively, $f(i) + g(i)$ and $f(i)g(i)$.

Definition Let f be a function whose domain is the set of integers, and define the function g by

$$g(i) = f(i) - f(i-1).$$

We say that f is a *convex* function if g is an increasing function; that is, f is convex if for all i,

$$f(i+1) - f(i) \geq f(i) - f(i-1).$$

Similarly, we say that f is a *concave* function if g is a decreasing function; that is, if for all i,

$$f(i+1) - f(i) \leq f(i) - f(i-1).$$