1. Preliminaries

1.1 Sets

A *set* is a collection of elements. If the set A consists of the n elements $a_1, a_2, \ldots, a_n$ then we express this by writing

$$A = \{a_1, a_2, \ldots, a_n\}.$$

Thus, for instance, the set consisting of all the integers between 6 and 10 is given by

$$B = \{6, 7, 8, 9, 10\}.$$

A set can be defined either by specifying all its elements, as just shown, or by specifying a defining property for its elements. Thus, the set B could have been defined as

$$B = \{\text{integers } i : |8 - i| \leq 2\}.$$

That is, B could have been defined as the set of all integers i such that the distance between i and 8 is less than or equal to 2.

A set consisting of a finite number of elements is said to be a *finite set*, whereas one consisting of an infinite number of elements is said to be an *infinite set*. The set $\mathcal{N}$ of all the nonnegative integers is an example of an infinite set. It is convenient to define the set that does not consist of any elements; we call this the *null* set and denote it by $\emptyset$.

We use the notation $a \in A$ to indicate that a is an element of A, and we use the notation $a \notin A$ to indicate that a is not a member of A.

Example 1.1a Let S be the set of all possible outcomes when a pair of dice are rolled. By an "outcome" we mean the pair (i, j), where i is the number of the side on which the first die lands and j is the number of the side for the second die. Then, the set of all outcomes that result in the sum of the dice being equal to 7 can be expressed as

$$S_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

or, alternatively, as

$$S_7 = \{(i, j) \in S : i + j = 7\}. \quad \square$$

If every element of A is also an element of B, then we say that A is a *subset* of B and write $A \subset B$. By this definition, every set is a subset of itself and hence, for example, $A \subset A$. Also, since there are no elements in the null set, it follows that every element of $\emptyset$ is also an element of A; thus, $\emptyset$ is a subset of every other set. If $A \subset B$ and $B \subset A$ then we may write $A = B$. That is, the sets A and B are said to be equal if every element of A is in B and every element of B is in A.

If A and B are sets then we define the new set $A \cup B$, called the *union* of A and B, to consist of all elements that are in A or in B (or in both). Also, we define the *intersection* of A and B, written either as $A \cap B$ or just AB, to consist of all elements that are in both A and B.

Example 1.1b In Example 1.1a, if A is the set of all outcomes for which the sum of the dice is 5 and if B is the set of outcomes for which the value of the second die exceeds that of the first die by the amount 3, then we have

$$A = \{(1, 4), (2, 3), (3, 2), (4, 1)\} \quad \text{and} \quad B = \{(1, 4), (2, 5), (3, 6)\};$$

also,

$$A \cup B = \{(1, 4), (2, 3), (3, 2), (4, 1), (2, 5), (3, 6)\} \quad \text{and}$$
$$AB = \{(1, 4)\}.$$

If we define C to be the set of all outcomes whose sum is equal to 6, then $AC = \emptyset$ because there are no outcomes whose sum is both 5 and 6. $\square$

A set is said to be a *universal set* if it contains (as subsets) all other sets under consideration. Let $\mathcal{U}$ be a universal set. For any set A, the set A^c, called the *complement* of A, is defined to be the set containing all the elements of the universal set $\mathcal{U}$ that are *not* in A.

Venn diagrams are often used to graphically represent sets. The universal set $\mathcal{U}$ is represented as consisting of all the points in a large rectangle, and sets are represented as consisting of all the points in circles within the rectangle. Particular sets of interest are indicated by shading

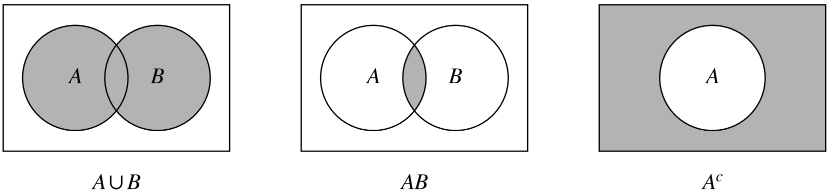

Figure 1.1

appropriate regions of the diagram. For instance, the Venn diagrams of Figure 1.1 indicate the sets $A \cup B$, AB, and A^c.

The operation of forming unions and intersections of sets obey certain rules that are similar to the rules of algebra. We list a few of them as follows:

Commutative laws: $A \cup B = B \cup A$, $AB = BA$.
Associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$, $(AB)C = A(BC)$.
Distributive laws: $(A \cup B)C = AC \cup BC$, $AB \cup C = (A \cup C)(B \cup C)$.

These relations are verified by showing that any element that is contained in the left-hand set is also contained in the right-hand one, and vice versa. For instance, to prove that

$$(A \cup B)C = AC \cup BC,$$

note that if $x \in (A \cup B)C$ then $x \in C$ and x is also in either A or B. If $x \in A$, then it is in AC and so is in $AC \cup BC$; similarly, if $x \in B$, then it is in BC and so is in $AC \cup BC$. Thus, $x \in AC \cup BC$, showing that

$$(A \cup B)C \subset AC \cup BC.$$

To go the other way, suppose that $y \in AC \cup BC$. Then y is either in both A and C or in both B and C. Therefore, we can conclude that y is in C and is in at least one of the sets A and B. But this means that $y \in (A \cup B)C$, showing that

$$AC \cup BC \subset (A \cup B)C,$$

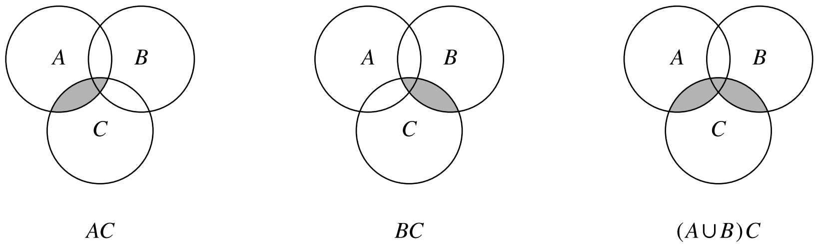

Figure 1.2

and the verification is complete. (The result could also be shown by using Venn diagrams; see Figure 1.2.)

We also define the intersection and union of more than two sets. Specifically, for sets $A_1, \ldots, A_n$ we define $\bigcup_{i=1}^n A_i$, the union of these sets, to consist of all elements that are in A_1, or in A_2, or in $A_3, \ldots$, or in A_n; that is, $\bigcup_{i=1}^n A_i$ is the set of all elements that are in at least one of the sets A_i, $i = 1, \ldots, n$. Similarly, we define $\bigcap_{i=1}^n A_i$, the intersection of these sets, to consist of all elements that are in each of the sets A_i, $i = 1, \ldots, n$.

1.2 Summation

If we let s be the sum of the four numbers x_1, x_2, x_3, x_4 then we can write

$$s = x_1 + x_2 + x_3 + x_4.$$

More compactly, we can use the summation notation $\sum$. Using this latter notation, we write

$$s = \sum_{i=1}^{4} x_i,$$

which means that s is equal to the sum of the x_i values as i ranges from 1 to 4. More generally, for $j \leq n$, we use the notation

$$s = \sum_{i=j}^{n} x_i$$

to mean that

$$s = x_j + x_{j+1} + \cdots + x_n.$$

Example 1.2a If $x_i = i^2$, find $\sum_{i=3}^{6} x_i$.

Solution.

$$\sum_{i=3}^{6} x_i = x_3 + x_4 + x_5 + x_6 = 9 + 16 + 25 + 36 = 86. \qquad \square$$

If S is a specified set of integers, then we use the notation

$$\sum_{i \in S} x_i$$

to represent the sum of all the values x_i that have indices in S.

Example 1.2b If $S = \{2, 4, 6\}$ then

$$\sum_{i \in S} x_i = x_2 + x_4 + x_6. \qquad \square$$

Consider the sum $T = \sum_{i=0}^{2} x_{2+i}$. Because T is equal to $x_2 + x_3 + x_4$, it follows that we can also express T as $T = \sum_{j=2}^{4} x_j$. Therefore, we see that

$$\sum_{i=0}^{2} x_{2+i} = \sum_{j=2}^{4} x_j.$$

When equating the right-hand summation to the left, we say that we are making the change of variable $j = 2 + i$. That is, summing the values x_{2+i} as i ranges from 0 to 2 is the same as summing the values x_j as j ranges from 2 to 4.

Example 1.2c Making the change of variable $j = n - i$ in the summation $\sum_{i=0}^{n} x_{n-i}$ gives the equivalent sum $\sum x_j$ as j ranges between n and 0. That is,

$$\sum_{i=0}^{n} x_{n-i} = \sum_{j=0}^{n} x_j. \qquad \square$$

We are sometimes interested in numbers that are expressed in the form $x_{i,j}$, where i and j both take values in some region. A quantity that is often of interest is the following "double sum" $D = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j}$, where

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_{i,j} \right).$$

Now arrange the numbers $x_{i,j}$ $(i = 1, \ldots, n,\ j = 1, \ldots, m)$ in the following row–column array, which has the number $x_{i,j}$ in row i, column j.

$$\begin{array}{ccccccc}
x_{1,1} & x_{1,2} & x_{1,3} & \ldots & x_{1,j} & \ldots & x_{1,m} \\
x_{2,1} & x_{2,2} & x_{2,3} & \ldots & x_{2,j} & \ldots & x_{2,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{i,1} & x_{i,2} & x_{i,3} & \ldots & x_{i,j} & \ldots & x_{i,m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n,1} & x_{n,2} & x_{n,3} & \ldots & x_{n,j} & \ldots & x_{n,m}
\end{array}$$

Because $\sum_{j=1}^{m} x_{i,j}$ is just the sum of the m array elements in row i, it follows that the double sum D is equal to the sum of the row sums. In other words, D is equal to the sum of all the elements in the array. Since the sum of all the array values can also be obtained by adding all the column sums and since the sum of the values of column j is $\sum_{i=1}^{n} x_{i,j}$, we have the following result.

Proposition 1.2.1

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} = \sum_{j=1}^{m} \sum_{i=1}^{n} x_{i,j}.$$

A corollary of this proposition is the following useful result.

Corollary 1.2.1

$$\sum_{i=1}^{n} \sum_{j=1}^{i} x_{i,j} = \sum_{j=1}^{n} \sum_{i=j}^{n} x_{i,j}.$$

Proof. Consider data values $x_{i,j}$, where i and j both takes values from 1 to n and where $x_{i,j} = 0$ when $j > i$. Then apply Proposition 1.2.1. □

A pictorial proof of Corollary 1.2.1 is obtained by noting that its left-hand side,

$$\sum_{i=1}^{n}\sum_{j=1}^{i} x_{i,j} = \sum_{j=1}^{1} x_{1,j} + \sum_{j=1}^{2} x_{2,j} + \cdots + \sum_{j=1}^{n} x_{n,j},$$

is equal to the sum of all the row sums whereas the right-hand side,

$$\sum_{j=1}^{n}\sum_{i=j}^{n} x_{i,j} = \sum_{i=1}^{n} x_{i,1} + \sum_{i=2}^{n} x_{i,2} + \cdots + \sum_{i=n}^{n} x_{i,n},$$

is the sum of all the column sums in the following array.

$$\begin{array}{lllll} x_{1,1} & & & & \\ x_{2,1} & x_{2,2} & & & \\ x_{3,1} & x_{3,2} & x_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ x_{n,1} & x_{n,2} & x_{n,3} & \ldots & x_{n,n} \end{array}$$

Example 1.2d

$$\begin{aligned} \sum_{i=1}^{3}\sum_{j=1}^{i}(i-j) &= \sum_{j=1}^{1}(1-j) + \sum_{j=1}^{2}(2-j) + \sum_{j=1}^{3}(3-j) \\ &= 0+1+3 = 4, \\ \sum_{j=1}^{3}\sum_{i=j}^{3}(i-j) &= \sum_{i=1}^{3}(i-1) + \sum_{i=2}^{3}(i-2) + \sum_{i=3}^{3}(i-3) \\ &= 3+1+0 = 4. \end{aligned}$$

□

Since $x\sum_{j=1}^{m} y_j = \sum_{j=1}^{m} xy_j$, it follows that

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{m} y_j\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} x_i\right) y_j = \sum_{j=1}^{m}\sum_{i=1}^{n} x_i y_j.$$

Example 1.2e Expand $(x_1 + \cdots + x_n)^2$.

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Solution.

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j\right)$$
$$= \sum_i \sum_j x_i x_j$$
$$= \sum_i \left(x_i x_i + \sum_{j \neq i} x_i x_j\right)$$
$$= \sum_i x_i^2 + \sum_i \sum_{j \neq i} x_i x_j.$$

For instance, the preceding yields

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + x_1 x_2 + x_2 x_1 = x_1^2 + x_2^2 + 2x_1 x_2. \qquad \square$$

Similar to the notation for summations is our notation $\prod$ for products,

$$\prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_n.$$

Example 1.2f

$$\prod_{i=1}^{4} i = 1 \cdot 2 \cdot 3 \cdot 4 = 24. \qquad \square$$

1.3 Mathematical Induction

Suppose that we have an infinite collection of statements, denoted $S_1, S_2, \ldots,$ and that we want to prove that they are all true. A proof by *mathematical induction* is obtained in the following manner:

(i) first prove that S_1 is true;
(ii) then show that, for any n, whenever S_n is true then S_{n+1} is also true.

Once (i) and (ii) are established, then from (i) we know that S_1 is true; which implies by (ii) that S_2 is true; which implies that S_3 is true; and so on. Thus, it follows that all of the S_n are true.

We now illustrate the use of mathematical induction by a series of examples.

Example 1.3a Prove that there are 2^n subsets of a set consisting of n elements.

Solution. In order to prove this by mathematical induction, we must first prove it for $n = 1$. But this is immediate, for if the set consists of a single element (i.e., if the set is $\{s\}$) then it has the two subsets Ø and $\{s\}$, where Ø is the empty set. Thus, part (i) of the mathematical induction approach is shown. To show part (ii), *assume* that the result is true for all sets of size n (this is called the *induction hypothesis*) and then consider a set S of size $n+1$. Focus attention on one of the elements of S, call it s, and let S' denote the set consisting of the n other elements of S. Because every subset of S that does not contain s is a subset of S', it follows from the induction hypothesis that there are 2^n subsets of S that do not contain s. On the other hand, since any subset of S that contains s can be obtained by adding s to a subset of S', it also follows from the induction hypothesis that there are 2^n of these subsets. Thus the total number of subsets of S is

$$2^n + 2^n = 2^n(1+1) = 2^{n+1},$$

and the result is proved. □

Example 1.3b For integer n, which is larger: 2^n or n^2?

Solution. Let us try a few cases:

$$\begin{aligned}
2^1 &= 2, & 1^2 &= 1;\\
2^2 &= 4, & 2^2 &= 4;\\
2^3 &= 8, & 3^2 &= 9;\\
2^4 &= 16, & 4^2 &= 16;\\
2^5 &= 32, & 5^2 &= 25;\\
2^6 &= 64, & 6^2 &= 36.
\end{aligned}$$

Thus, based on this enumeration, a reasonable conjecture is that $2^n > n^2$ for all values of $n \geq 5$. To prove this, we start by showing it to be true when $n = 5$; this was demonstrated by our preceding calculations. So now assume that, for some n $(n \geq 5)$,

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$$2^n > n^2.$$

We must show that the preceding implies that $2^{n+1} > (n+1)^2$, which may be accomplished as follows.

First, note that

$$2^{n+1} = 2 \cdot 2^n > 2n^2,$$

where the inequality follows from the induction hypothesis. Hence, it will suffice to show that, for $n \geq 5$,

$$2n^2 \geq (n+1)^2$$

or (equivalently)

$$2n^2 \geq n^2 + 2n + 1$$

or

$$n^2 - 2n - 1 \geq 0$$

or

$$(n-1)^2 - 2 \geq 0$$

or

$$n - 1 \geq \sqrt{2},$$

which follows because $n \geq 5$. □

Example 1.3c Derive a simple expression for the following function:

$$f(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Solution. Again, let us begin by calculating the value of $f(n)$ for small values of n, hoping to discover a general pattern that we can then prove by mathematical induction. Such a calculation gives

$$f(1) = 1/2,$$
$$f(2) = 1/2 + 1/6 = 2/3,$$
$$f(3) = 2/3 + 1/12 = 3/4,$$
$$f(4) = 3/4 + 1/20 = 4/5.$$