Design Theory

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Contents

Contents of Volume I

Chapters IX–XIII plus the appendices can be found in Volume II. The complete bibliography and index for both volumes is included here.

IX

Recursive Constructions

Alabado sea la infinita
Urdimbre de los efectos y de las causas...
(Borges)

In the first few sections of this chapter we shall develop some important recursive construction methods which will be applied to mutually orthogonal Latin squares, pairwise balanced designs, and in particular block designs.

§1. Product Constructions

In this section, we consider some product constructions. The first two of these concern difference matrices and are due to Jungnickel (1979) and Shrikhande (1964), respectively; the simple proofs are left to the reader. We note that Shrikhande's result generalises Lemma I.9.6.

1.1 Lemma. *Let $A = (a_{ij})$ be a $(g, k; \lambda)$-difference matrix and $B = (b_{ij})$ an $(h, k; \mu)$-difference matrix over the (additively written) respective finite groups G and H. Then the $k \times \lambda\mu gh$-matrix*

$$\left(\begin{array}{c|c|c} (a_{11}, b_{11}) \ \ldots \ (a_{11}, b_{1n}) & (a_{12}, b_{11}) \ \ldots \ (a_{12}, b_{1n}) & \ldots\ldots(a_{1m}, b_{1n}) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ (a_{k1}, b_{k1}) \ \ldots \ (a_{k1}, b_{kn}) & (a_{k2}, b_{k1}) \ \ldots \ (a_{k2}, b_{kn}) & \ldots\ldots(a_{km}, b_{kn}) \end{array}\right)$$

is a $(gh, k; \lambda\mu)$-difference matrix over $G \oplus H$, where $m = \lambda g$ and $n = \mu h$. ■

1.2 Lemma. *Let A be an $OA_\lambda(k, g)$ with entries in a finite group G and let $D = (d_{ij})$ $(i = 1, \ldots, k', j = 1, \ldots, g\lambda')$ be a $(g, k'; \lambda')$-difference matrix*

over $G = \{\gamma_1, \ldots, \gamma_g\}$. *Then the matrix* $A \otimes D$ *defined by*

$$A \otimes D = \begin{pmatrix} a_{11} + D & \ldots\ldots & a_{1,g^2\lambda} + D \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{k1} + D & \ldots\ldots & a_{k,g^2\lambda} + D \end{pmatrix}$$

is an $OA_{\lambda\lambda'}(kk', gg')$.[1]

If E *is a* (g, k, λ)*-difference matrix, then* $E \otimes D$ *is a* $(g, kk', g\lambda\lambda')$*-difference matrix. In particular, the Kronecker product of two generalised Hadamard matrices* (VIII.3.4) *is a generalised Hadamard matrix.* ■

1.3 Example. The existence of (3, 3; 1)- and (3, 6; 2)-difference matrices (see Theorem VIII.3.14) implies that of a (3, 18; 6)-difference matrix over $\mathbb{Z}_3 \times \mathbb{Z}_6$. More generally, we obtain $(3, 3\lambda; \lambda)$-difference matrices for all $\lambda = 2^i 3^j$ with $j \geq i - 1$. All these difference matrices are generalised Hadamard matrices.

1.4 Definition. Let $(A, \circ)$, $(B, *)$ be quasigroups (see Definition VIII.4.10). Their *direct product* is defined as the set $A \times B$ with the operation $\square$ defined by

$$(a, b) \square (a', b') := (a \circ a', b * b'). \tag{1.4.a}$$

1.5 Lemma. *Let* $(A, \circ)$ *and* $(A, \circ')$ *be orthogonal quasigroups, and let* $(B, *)$, $(B, *')$ *be orthogonal quasigroups too. Then the direct products* $(A, \circ) \times (B, *)$ *and* $(A, \circ') \times (B, *')$ *are orthogonal quasigroups. If* $(A, \circ)$ *and* $(B, *)$ *are idempotent, then* $(A, \circ) \times (B, *)$ *is also idempotent.* ■

The proof is straightforward. Again MacNeish's theorem (I.7.7.b) follows immediately; furthermore

$$N^*(gh) \geq \min\{N^*(g), N^*(h)\}. \tag{1.5.a}$$

For the sake of completeness we mention the following connection between quasigroups and Steiner triple systems.

[1] Note that this would be the Kronecker product of matrices if G were written multiplicatively.

1.6 Observations. Define a binary operation $\circ$ on the point set V of an $S(2,3;v)$ by

(1.6.a) $a \circ b = c$ if $a \neq b$ and $\{a, b, c\}$ is a line,

(1.6.b) $a \circ a = a$ for all $a \in V$.

Then $(V, \circ)$ is an idempotent quasigroup with the properties

(1.6.c) $x \circ y = y \circ x$,

(1.6.d) $x \circ (x \circ y) = y$ for all $x, y \in V$.

Conversely, every finite idempotent quasigroup satisfying (1.6.c) and (1.6.d) determines an *STS* by the rule: x, y, z are on a line iff $x \neq y$ and $x \circ y = z$.

1.7 Lemma. *Let* $(V, \circ)$ *and* $(W, *)$ *be idempotent quasigroups which satisfy* (1.6.c, d)*. Then* $(V, \circ) \times (W, *)$ *has the same properties.* ■

The proof is straightforward.

1.8 Corollary.

(1.8.a) $u, v \in B(3) \Rightarrow uv \in B(3)$. ■

Next we mention analogous properties of Steiner quadruple systems.

1.9 Observations. Let an *SQS* be given. Define a ternary operation f on the point set V by

(1.9.a) $f(x, y, z) := u$ if $|\{x, y, z\}| = 3$ and $\{x, y, z, u\}$ is a block;

(1.9.b) $f(x, x, y) = f(x, y, x) = f(y, x, x) := y$ for all $x, y \in V$.

Then f has the properties

(1.9.c) $f(x, y, z) = f(y, x, z) = f(x, z, y)$,

(1.9.d) $f(x, y, f(x, y, z)) = z$ for all $x, y, z \in V$,

and the equation

(1.9.e) $f(a, b, x) = c$

has a unique solution x for all $a, b, c \in V$. Conversely, every ternary algebra with these three properties determines an *SQS* by the following rule. Four

distinct points a, b, c, d are on a block iff $f(a, b, c) = d$. If (V, f) and (W, g) are such ternary algebras, then their direct product, defined in the obvious way, has the same properties. Hence

(1.9.f) $\quad u, v \in S(3,4) \Rightarrow uv \in S(3,4).$

Note that $2 \in S(3,4)$ (two points, no blocks), and that a ternary operation $\circ$ on a 2-set, satisfying (1.9.b), has all the properties of 1.9. Hence (1.9.f) specializes to

(1.9.g) $\quad 2 \cdot S(3,4) \subseteq S(3,4).$

In view of Hanani's theorem stating that $S(3,4) = 2\,\mathbb{N} \setminus 6\,\mathbb{N}$, (1.9.f) is a very weak result; see §10 for a proof of Hanani's theorem.

1.10 Remark. These examples may be generalised to the idea of applying universal algebra to combinatorial structures; cf. Ganter (1976a), Evans (1975), Quackenbush (1975). A nice introductory paper on this subject was given by Evans (1979).

Next we shall use difference matrices and orthogonal arrays for product constructions of difference families, as well as the other way round.

1.11 Proposition. *Let k be the order of an affine plane. Furthermore, let $A = (a_{ij})$ be an $OA(k,k)$ over $S = \mathbb{N}_1^k$. Moreover, let $\boldsymbol{D} = (D_1, \ldots, D_s)$ and $\boldsymbol{E} = (E_1, \ldots, E_t)$ be difference families in the groups G and G' with parameters (v, k, λ) and (v', k, λ'), respectively, say $D_i = \{d_{i1}, \ldots, d_{ik}\}$ and $E_j = \{e_{j1}, \ldots, e_{jk}\}$ for $i \in \mathbb{N}_1^s$ and $j \in \mathbb{N}_1^t$. Then there is a $(vv', k, \lambda\lambda')$-difference family in $G \oplus G'$.*

Proof. We may assume that the last k columns of A are $(1, 1, \ldots, 1)^T, \ldots, (k, k, \ldots, k)^T$. We omit these k columns and get a $k \times k(k-1)$-matrix $B = (b_{ij})$. Now let the desired difference family $\boldsymbol{F}$ consist of the following base blocks:

λ' copies of $\{(d_{i1}, 0), \ldots, (d_{ik}, 0)\}$ for each $i \in \mathbb{N}_1^s$;
λ copies of $\{(0, e_{j1})\}, \ldots, (0, e_{jk})\}$ for each $j \in \mathbb{N}_1^t$;
one copy of $\{(d_{i1}, e_{jb_{1n}}), \ldots, (d_{ik}, e_{jb_{kn}})\}$ for each $n \in \mathbb{N}_1^{k^2-k}$ and each pair $(i, j) \in \mathbb{N}_1^s \times \mathbb{N}_1^t$.

The somewhat lengthy though not difficult verification of this construction is left to the reader; cf. Jungnickel (1978). ■

1.12 Corollary. *With notation as in* VII.5.3, *for prime powers k we have*

(1.12.a) $v \in D_\lambda(k)$ *and* $v' \in D_{\lambda'}(k) \Rightarrow vv' \in D_{\lambda\lambda'}(k)$;

in particular

(1.12.b) $v, v' \in D(k) \Rightarrow vv' \in D(k)$. ■

1.13 Examples. (a) Let q and $q+1$ be prime powers and $n \in \mathbb{N}$. Then there is a $((q^2+q+1)^n, q+1, 1)$-difference family in $(\mathbb{Z}_{q^2+q+1})^n$. The case $n = 1$ is just a Singer difference set, see Theorem VI.1.10. Note that the result also follows from Lemma VII.5.5 if $q^2 + q + 1$ is also a prime power (e.g. for $q = 2, 3, 8$), but not for $q = 4, 7, 16, 31, 127$.

(b) Let $v = q_1 \dots q_n$ be the prime power factorisation of v and assume

$$q_i \equiv 1 \pmod 6 \quad \text{for } i = 1, \dots, n.$$

Then there is a $(v, 3, 1)$-difference family in $EA(q_1) \oplus \dots \oplus EA(q_n)$, by (VII.5.4.a) and Proposition 1.11.

(c) Let k be a prime power and assume that each factor q in the prime power factorisation of v is congruent to $1 \bmod k(k-1)$ and sufficiently large. Then, by Wilson's Theorem VII.6.6 and Corollary 1.12, there is a $(v, k, 1)$-difference family. The existence of difference families for given k, λ and all sufficiently large v satisfying the necessary condition (VII.1.9.b) is an unsettled question.

1.14 Proposition. *Assume the existence of a* (v, k, λ)*-difference family* $\boldsymbol{D} = (D_1, \dots, D_s)$ *in G and of both a* $(g, k, \lambda\lambda')$*-difference family* $\boldsymbol{E} = (E_1, \dots, E_t)$ *and a* (g, k, λ')*-difference matrix* $A = (a_{ij})$ *in G′. Then there is a* $(gv, k, \lambda\lambda')$*-difference family* $\boldsymbol{F}$ *in* $G \oplus G'$.

Proof. As base blocks of $\boldsymbol{F}$ take

$$\{0\} \times E_j \quad \text{for } j \in \mathbb{N}_1^t$$

and

$$\{(d_{i1}, a_{1j}), \dots, (d_{ik}, a_{kj})\} \quad \text{for all } (i, j) \in \mathbb{N}_1^s \times \mathbb{N}_1^{g\lambda'},$$

where $D_i = \{d_{i1}, \dots, d_{ik}\}$. Then it is easily checked that $\boldsymbol{F}$ is a $(gv, k, \lambda\lambda')$-difference family; cf. Jungnickel (1978). ■

1.15 Examples. (a) Proposition 1.14 and Lemma VIII.3.10 yield alternative proofs for Examples 1.13.(b), (c). An alternative proof of Example 1.13.(a) will be given a little later.

(b) Using Proposition 1.14, Lemma VIII.3.10 and Wilson's Theorem VII.6.6, we obtain the existence of a (v, k, λ)-difference family whenever v admits a prime power factorisation

(1.15.a) $\quad v = q_1 q_2 \dots q_n$

with $\lambda(q_i - 1) \equiv 0 \pmod{k(k-1)}$, where the q_i $(i = 1, \dots, n)$ are sufficiently large.

(c) Assume that k or $k-1$ divides 2λ. Then there exists a (v, k, λ)-difference family whenever each factor q_i in the prime power factorisation of v satisfies the condition

(1.15.b) $\quad \lambda(q_i - 1) \equiv 0 \pmod{k(k-1)}$.

This follows from Lemma 1.1 (with $\lambda' = 1$), together with Lemma VIII.3.10. Thus, for example, we obtain (compare with Examples VII.5.4)

(1.15.c) $\quad v \in D_1(3) \cap D_2(4) \cap D_5(6) \cap D_6(7)$,

whenever $q_i \equiv 1 \pmod 6$ for $i = 1, \dots, m$;

(1.15.d) $\quad v \in D_2(5) \cap D_3(6)$,

whenever $q_i \equiv 1 \pmod{10}$ for $i = 1, \dots, n$;

(1.15.e) $\quad v \in D_3(7) \cap D_4(8)$,

whenever $q_i \equiv 1 \pmod{14}$ for $i = 1, \dots, n$;

(1.15.f) $\quad v \in D_4(9)$,

whenever $q_i \equiv 1 \pmod{18}$ for $i = 1, \dots, n$.

(d) Let $t, n \in \mathbb{N}$ and $4t - 1$ be a prime power. Then

(1.15.g) $\quad (4t-1)^n \in D_{t-1}(2t-1)$.

This result follows by using a Paley difference set (Theorem VI.1.12), and is a special case of (c).

(e) There is a $(91, 10, 1)$-difference set in $\mathbb{Z}_{91}$ (Theorem VI.1.10). Also there are $(13, 13; 1)$-difference matrices (Lemma VIII.3.10), and $(7, 14; 2)$-difference

matrices (Theorem VIII.3.14). Thus there is a $(91, 10; 2)$-difference matrix in $\mathbb{Z}_{91}$ by Lemma 1.1. Hence by an easy induction,

(1.15.h) $\quad 91^n \in D_{2^{n-1}}(10)$ for every $n \in \mathbb{N}$.

1.16 Exercises. (a) Show the existence of $(133^n, 12, 2^{n-1})$-difference families and of $(16^m 31^n, 6, 2^m)$-difference families $(m, n \in \mathbb{N})$.

(b) Assume that $q^d + \ldots + q + 1$ is a prime power. Show that, for $n \in \mathbb{N}$,

$(q^d + \ldots + q + 1)^n \in D_\lambda(q^{d-1} + \ldots + q + 1)$ with $\lambda = q^{d-2} + \ldots + q + 1$,
$(q^{d+1} + \ldots + q + 1)^n \in D_\lambda(q^d + \ldots + q + 1)$ with $\lambda = q^{d-1} + \ldots + q + 1$.

In a certain sense, Propositions 1.11 and 1.14 are complementary. In 1.11 one uses information on k and in 1.14 information on g. Proposition 1.11 has the advantage of allowing the combination of (v, k, λ)-difference families for $\lambda > 1$ without enlarging the λ-value. In Proposition 1.14 we used difference matrices to construct difference families. The converse is done in the following construction; cf. Jungnickel (1979).

1.17 Proposition. *Assume the existence of a (v, g, λ)-difference family over G and of a $TD[k; g]$ with a parallel class. Then there exists a $(v, k; \lambda)$-difference matrix over G.*

Proof. The existence of a parallel class allows us to assume that $(1, \ldots, 1)^T$, $\ldots$, $(g, \ldots, g)^T$ are columns of an $OA(g, k; 1)$ corresponding to the given $TD[k; g]$. As in the proof of Proposition 1.11, we drop these columns and obtain a $(k \times g(g-1))$-matrix $B = (b_{mn})$ over $\mathbb{N}_1^g$. Let $D_i = (d_{i1}, \ldots, d_{ig})$ be the i-th base block in the given (v, g, λ)-difference family $\boldsymbol{D} = (D_1, \ldots, D_s)$ and replace each entry b_{mn} of B by $d_{ib_{mn}}$. This yields a $k \times g(g-1)$-matrix A_i. Put

$$A := (A_1 A_2 \ldots A_s 0),$$

where 0 denotes a $k \times \lambda$-zero matrix. Now consider rows h and l of A. The difference 0 occurs λ times from the zero matrix used. As the submatrix of B determined by rows h and l contains in its columns each pair with distinct entries precisely once, we obtain the difference $d_{i\beta} - d_{i\gamma}\,(\beta \neq \gamma)$ from A_i exactly once, by our construction. As $\boldsymbol{D}$ was a (v, g, λ)-difference family, $(A_1 A_2 \ldots A_n)$ yields each non-zero element of G exactly λ times from rows h and l. ■

Note that $s = \frac{b}{v} = \lambda\frac{v-1}{g(g-1)}$, hence the column number of A is $sg(g-1)+\lambda = v$, as it must be.

1.18 Examples. (a) Let q be a prime power and assume the existence of a $TD[k; q+1]$ with a parallel class. Then there exists a $(q^2+q+1, k; 1)$-difference matrix in $\mathbb{Z}_{q^2+q+1}$ by Singer's Theorem VI.1.10.

(b) There are $(15, 7; 3)$-, $(21, 5; 1)$-, $(40, 13; 4)$-, and $(57, 8; 1)$-difference matrices in the cyclic groups of the respective orders. The reader is asked to check this.

(c) Let q be a prime power and let G be a group of order $q+1$. Then there is a $(q+1, q; q-1)$-difference matrix over G. To see this, use the trivial $(q+1, q, q-1)$-difference set $G \setminus \{0\}$. Note that this yields non-abelian difference matrices too (though of moderate size; the maximum feasible value of k is q^2-1 by Corollary VIII.3.7).

It is worthwile to state the following consequence of Proposition 1.17.

1.19 Corollary. *Assume the existence of a $(v, g, 1)$-difference family. Then*

(1.19.a) $\quad g \in TD(k) \Rightarrow v \in TD(k)$.

In particular,

(1.19.b) $\quad q+1 \in TD(k) \Rightarrow q^2+q+1 \in TD(k)$ *for prime powers q.*

Thus

(1.19.c) $\quad 21 \in TD(6)$,

(1.19.d) $\quad 57 \in TD(9)$,

(1.19.e) $\quad 273 \in TD(18)$.

Proof. Use Lemma 1.1 together with (I.7.15.a) and Corollary VIII.3.8. The particular examples follow from 1.18.(a). ■

The values given in (1.19.c, d, e) improve those following from MacNeish's Corollary I.7.8 and give the best result for the numbers 57 and 273 known to date. For 21, the better result $21 \in TD(7)$ is known, see Corollary VIII.3.17.

1.20 Exercises. (a) Try to generalise Proposition 1.17 by using a $TD_\mu[k; g]$ with a parallel class. Show that this is possible under the additional assumption $k(\mu - 1) \leq \lambda\mu$. Use this to show the existence of $(q + 1, q^2; q^2 - q)$- and of $(q+1, 2q; 2(q-1))$-difference matrices in any group of order $q+1$, whenever q is a prime power.

(b) Let G be any group of order g and assume the existence of a $TD_\lambda[k; g]$. Show the existence of a $(g, k; g\lambda)$-difference matrix over G.

Hint: Construct the corresponding $OA_\lambda(k, g)$ on the symbol set G.

(c) Prove the following strengthening of the construction given in Example 1.18.(c) which is due to Colbourn and Kreher (1996): Assume the existence of an $OA_\lambda(k, g)$ with at least λ constant columns. Then, over any group of order $g + 1$, a $(g + 1, k; \lambda(g - 1))$-difference matrix exists. In particular, over any group of order $g + 1$, a $(q + 1, q + 1; q - 1)$-difference matrix exists provided that q is a prime power, cf. Jungnickel (1995b).

1.21 Remarks. (a) A recent study of quasigroups, their connections to various types of designs and the corresponding spectra is due Bennett (1989) who also has an extensive bibliography on this area of research. Regarding the application of universal algebra to designs, we mention two further interesting papers, namely Ganter and Metz (1977) and Ganter and Werner (1975).

(b) Further recursive constructions for cyclic difference families and cyclic designs were given by Colbourn and Mathon (1980), Colbourn and Colbourn (1980b, 1984), Jimbo and Kuriki (1983), Grannell and Griggs (1986), Mathon (1987), Jimbo (1993) and Buratti (1997a, c, 1998c). Narayani and Blanchard (1995) gave a new composition theorem for (in general non-cyclic) difference families; a generalisation of their approach can be found in Ray-Chaudhuri and Zhu (1992).

(c) Regarding cyclic Steiner systems with $t \geq 3$ (cf. §III.9), the case of cyclic Steiner quadruple systems has found particular interest, see §VIII.10.

(d) Evans (1989) gave a partial converse of the construction of a cyclic $(q^2 + q + 1, k; 1)$-difference matrix from a cyclic Singer difference set and a $TD[k; q + 1]$ mentioned in Example 1.18.

(e) Colbourn and Kreher (1996) gave some new recursive constructions for difference matrices with $\lambda \neq 1$, using *PBD*'s, *OA*'s and finite fields as ingredients. They also provided a table on the largest known size of an (s, r, λ)-difference matrix in the range $s \leq 32$ and $\lambda \leq 30$. For a larger table of this kind, see

§IV.11.4 of Colbourn and Dinitz (1996a). Further recursive constructions for difference matrices and difference families are due to Buratti (1998a).

In this section we have seen how closely related the concepts of difference families and difference matrices are. The corresponding interaction between pairwise balanced designs and transversal designs will be of paramount importance in the recursive constructions of designs.

§2. Use of Pairwise Balanced Designs

We shall present some important recursive constructions, in particular for *PBD*'s and *TD*'s. We use Hanani's notation (cf. I.2.19).

2.1 Lemma. *Let $g \in TD(k+n)$ and $k, k+1, \ldots, k+n \in K$; also, let $0 \le g_i \le g$ for $i = 1, \ldots, n$. Then*

$$kg + g_1 + \ldots + g_n \in GD(K, \{g, g_1, \ldots, g_n\}) \subseteq B(K \cup \{g, g_1, \ldots, g_n\}). \tag{2.1.a}$$

Proof. Delete $g - g_1, \ldots, g - g_n$ points from the last n point classes of a $TD[k+n; g]$. ■

2.2 Lemma. *Let $g \in TD(k+n)$ and $k, k+1, \ldots, k+n, g+1, g_1+1, \ldots, g_n+1 \in K$; $0 < g_i \le g$ for $i = 1, \ldots, n$. Then*

$$kg + g_1 + \ldots + g_n + 1 \in B(K). \tag{2.2.a}$$

Proof. Proceed as in Lemma 2.1 and use (I.6.5.b). ■

2.3 Theorem. *If $v \in B(L)$ and $L \subseteq B(K)$, then $v \in B(K)$.*

Proof. Let $\boldsymbol{D} = (V, \{B_1, \ldots, B_b\}, \in)$ be an $S(2, L; v)$. By hypothesis, there are *PBD*'s $\boldsymbol{D}_i = (B_i, \{C_{i1}, \ldots, C_{ib_i}\}, \in)$ with $|C_{i\mu}| \in K$ for all i, μ. Then

$$\left(V, \left\{C_{ij} : i \in \mathbb{N}_1^b,\ j \in \mathbb{N}_1^{b_i}\right\}, \in\right)$$

is the desired *PBD*. Wilson (1972a) calls this procedure "breaking up blocks". ■

2.4 Corollary.

(2.4.a) $B(B(K)) = B(K)$ *for each* $K \subseteq \mathbb{N}$.

Note that $B(\emptyset) = B(\{1\}) = \{1\}$, *and that the usual convention* $1 \in B(K)$ *for all* $K \subseteq \mathbb{N}$ *is reasonable.*

Proof. Since $(A, \{A\}, \in)$ is a trivial $S(2, L, |A|)$ for any finite set A with $|A| \in L$, we have $L \subseteq B(L)$ for each $L \subseteq \mathbb{N}$; in particular $B(K) \subseteq B(B(K))$. By Theorem 2.3, we also have $B(B(K)) \subseteq B(K)$. ■

2.5 Lemma. *If* $v \in S_\lambda(t, K)$ *and* $K \subseteq S_\mu(t, L)$ *(for notation see* I.3.3*), then* $v \in S_{\lambda\mu}(t, L)$. *Similarly,*

(2.5.a) $GD_\lambda(K, M) = GD_\lambda(B(K), M)$. ■

The proofs are straightforward generalisations of the previous one. Note that K and L may be infinite subsets of $\mathbb{N}$.

2.6 Lemma.

(2.6.a) $GD_\lambda(K, B(K, \lambda)) \subseteq B(K, \lambda)$,

(2.6.b) $GD_\lambda(K, B(K, \lambda) - 1) + 1 \subseteq B(K, \lambda)$,

(2.6.c) $GD_\lambda(K, GD_\lambda(K, N)) \subseteq GD_\lambda(K, N)$.

Proof. In a $GD_\lambda[K, \mathrm{GD}_\lambda, (K, N); v]$, say $(V, \boldsymbol{B}, I)$, form a $GD_\lambda[K, N; |G|]$ on each point class G, say $(G, \boldsymbol{A}_G, I)$. Then

$$\left(V, \boldsymbol{B} + \sum_G \boldsymbol{A}_G, I\right)$$

is a $GD_\lambda[K, N; v]$. This proves (2.6.c). (2.6.a) is a special case. In order to prove (2.6.b), introduce a new point ∞ on the point classes G and form an $S_\lambda(2, K, |G| + 1)$ on each set $G \cup \{\infty\}$. This proves (2.6.b). ■

2.7 Observation. The mapping $B : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ which is defined by

(2.7.a) $K \mapsto B(K)$, for all $K \subseteq \mathbb{N}$,

is a *closure operator*; i.e. it has the properties

(2.7.b) $K \subseteq B(K)$,

(2.7.c) $K \subseteq B(L) \Rightarrow B(K) \subseteq B(L)$,

and hence

(2.7.d) $B(B(L)) = B(L)$

for all $K, L \subseteq \mathbb{N}$.

A subset $K \subseteq \mathbb{N}$ is called *closed* if $B(K) = K$ or, in case of ambiguity, *B-closed*.

The concept of closed subsets of $\mathbb{N}$ was introduced by Wilson (1972a, b). It is a very important tool which considerably simplified previous constructions of Hanani and other authors.

2.8 Examples. (a) For each $K \subseteq \mathbb{N}$ and $\lambda \in \mathbb{N}$, the set $B(K, \lambda)$ is closed.

(b) The sets $6\mathbb{N} + \{1, 3\}$, $12\mathbb{N} + \{1, 4\}$, $20\mathbb{N} + \{1, 5\}$ are closed. The next lemma generalises this example.

2.9 Lemma. *Let $K \subseteq \mathbb{N}$ be given and define α, β by*

(2.9.a) $\alpha := \gcd\{k - 1 : k \in K\}$,

(2.9.b) $\beta := \gcd\{k(k-1) : k \in K\}$.

Then

(2.9.c) $\lambda(v - 1) \equiv 0 \pmod{\alpha}$,

(2.9.d) $\lambda v(v - 1) \equiv 0 \pmod{\beta}$

are necessary conditions for the existence of an $S_\lambda(2, K; v)$. Let L be the set of all $v \in \mathbb{N}$ satisfying these two conditions. Then L is closed.

Proof. The necessity of (2.9.c) and (2.9.d) follows by counting the flags (p, B) with $p \not\in C$ and $c, p\ I\ B$ for a given point c, and the triples (x, y, B) with $x \neq y$ and $x, y\ I\ B$. Note that Corollary I.2.11 is the special case $K = \{k\}$.

In order to prove that L is closed, let $\boldsymbol{D} = (V, \boldsymbol{B}, \in)$ be an $S(2, L; v)$. Choose a point $d \in V$ and count the flags (p, B) with $p \neq d$ and $d, p\ I\ B$. Thus

$$v - 1 = \sum_{B \in (d)} (|B| - 1),$$
$$\lambda(v-1) = \sum_{B \in (d)} \lambda(|B| - 1) \equiv 0 \,(\mathrm{mod}\, \alpha),$$

by (2.9.c). Now count the triples (x, y, B) with $x \neq y$ and $x, y\ I\ B$. Thus

$$v(v-1) = \sum_{B \in \boldsymbol{B}} |B|(|B| - 1),$$
$$\lambda v(v-1) = \sum_{B \in \boldsymbol{B}} \lambda|B|(|\boldsymbol{B}| - 1) \equiv 0 \,(\mathrm{mod}\, \beta),$$

by (2.9.d), since $|B| \in L$. Hence $v \in L$. ■

As an example consider the set $30\,\mathbb{N} + \{1, 6, 16, 21\}$. Note that $30\,\mathbb{N} + \{1, 6\}$ is not closed; cf. Exercise VIII.1.9.

2.10 Definition. Let $TD^*(k)$ be the set of $g \in \mathbb{N}$ for which a $TD[k; g]$ with a parallel class exists. By Lemma VIII.4.13, $TD^*(k)$ is also the set of $g \in \mathbb{N}$ for which $k - 2$ mutually orthogonal idempotent quasigroups of order g exist. The following theorem is a special case of a theorem of Bose, Shrikhande and Parker (1960); see also Theorem X.1.1.

2.11 Theorem. *The set $TD^*(k)$ is closed for every $k \in \mathbb{N}$.*

Proof. It suffices to prove the following assertion. Let $(V, \boldsymbol{B}, \in)$ be a *PBD*. Assume that on each block $B \in \boldsymbol{B}$ there are two idempotent orthogonal quasigroups $(B, \overset{B}{\circ})$, $(B, \overset{B}{*})$. Define $\circ$ and $*$ on V as follows.

(2.11.a) $\quad x \circ x = x * x = x \quad$ for all $x \in V$

and

(2.11.b) $\quad x \circ y := x \overset{B}{\circ} y, \quad x * y := x \overset{B}{*} y,$

if $x \neq y$ and B is the block through x and y. Then $(V, \circ)$, $(V, *)$ are orthogonal idempotent quasigroups. The proof is left to the reader.

Another proof of the theorem will be given in §X.1. ■

2.12 Examples. We use Theorem 2.11 and Lemma 2.1 with $n = 1, 2$. Thus

$$\begin{aligned} 50 &= 7 \cdot 7 + 1 \in B(\{7, 8\}) \subseteq TD^*(7), \\ 54 &= 7 \cdot 7 + 5 \in B(\{5, 7, 8\}) \subseteq TD^*(5), \\ 70 &= 7 \cdot 9 + 7 \in B(\{7, 8, 9\}) \subseteq TD^*(7), \\ 57 &\in B(8) \subseteq TD^*(8), \\ 253 &\in B(\{16, 13\}) \subseteq TD^*(11), \text{ cf. (VIII.5.6.b) with } q = 4. \end{aligned}$$

2.13 Proposition.

(2.13.a) $\quad B(\{3, 4\}) = (3\,\mathbb{N}_0 + \{1, 3\}) \setminus \{6\}.$

Proof. $6 \notin B(\{3, 4\})$ follows from (I.8.4.a). "$\subseteq$" now follows from (2.9.c, d).

If $g \in K := B(\{3, 4\})$ and $g \in TD(4)$, $g_1 \in K \cup \{0\}$, and $g_1 \leq g$, then $3g + g_1 \in K$. This follows with Lemma 2.1, since K is closed. Note that $7 \in B(3) \subset K$, and that $g \in TD(4)$ if $g \not\equiv 2 \pmod 4$, by (I.7.8.a). The table

g	3	4	7	9	12	13
g_1	0, 1, 3	0, 1, 3, 4	0, 1, 3, 4	0, 1, 3, 4, 7	0, 1, 3, 4, 7, 9	0, 1, 3, 4, 7, . . . , 13
$3g + g_1$	9, 10, 12	12, 13, 15, 16	21, . . . , 25	27, . . . , 31, 34	36, . . . , 43, 45	39, . . . , 43, 46, . . . , 52

yields some small values of K, but leaves the gaps 18, 19, and 33. But $19, 33 \in K$ by (I.6.11.a), and $18 \in K$ by (I.5.13.a). Now it is easily seen that each $x \equiv 0$ or $1 \pmod 3$ with $x > 52$ has a representation $x = 3g + g_1$ with $g, g_1 \equiv 0$ or $1 \pmod 3$, $g \not\equiv 2 \pmod 4$ and $7 \leq g_1 \leq g$. Hence the assertion follows by induction. ■

The proof would be even shorter if one used result (I.7.9.b).

§3. Applications of Divisible Designs

3.1 Definition. A $GD_\lambda[K, G]$ is a divisible design $\boldsymbol{D}$ with parameter λ, block sizes in $K \subseteq \mathbb{N}$ and point class sizes in G, and a $GD_\lambda[K]$ is a $GD_\lambda[K, \mathbb{N}]$. Without loss of generality, we may assume that the point set is $\{1, 2, \ldots, v\} \subset \mathbb{N}$, and that the (non-empty) point classes are $G_1, G_2, \ldots, G_s$, where

(3.1.a) $\quad x \in G_i, \quad y \in G_j \quad \text{and} \quad i < j \quad \Rightarrow \quad x < y.$

The s-tuple $(G_1, \ldots, G_s)$ is called the *point class list* (or, briefly, the *class list*) of $\boldsymbol{D}$. We put $g_i := |G_i|$ for $i = 1, \ldots, s$, hence $\sum_{i=1}^{s} g_i = v$. The *point class type* (or, briefly, the *class type*) of $\boldsymbol{D}$ is the s-tuple

(3.1.b) $\qquad (g_1, \ldots, g_s) \in \mathbb{N}^s.$

If some of the g_i $(i = 1, \ldots, s)$ are equal, say

(3.1.c) $\qquad g_1 = \ldots = g_{\alpha_1} \neq g_{\alpha_1+1} = \ldots = g_{\alpha_1+\alpha_2} \neq g_{\alpha_1+\alpha_2+1} = \ldots,$

we also write

(3.1.d) $\qquad \left(g_1^{\alpha_1}, \ldots, g_t^{\alpha_t}\right) \quad \text{with} \quad \sum_{j=1}^{t} \alpha_j = s$

instead of (3.1.b). Thus every pairwise balanced design has class type (1^v), and every $GD_\lambda[K, g; sg]$ has class type (g^s).

If $\boldsymbol{D}$ is an incidence structure (in most applications a *PBD* or a *GDD*) on the point set $V = \{1, 2, \ldots, v\}$, then a *weighting* of V is just a mapping $w: V \to \mathbb{N}_0$.

With these conventions, we can state the following general composition theorem due to Wilson (1972b).

3.2 Theorem. *Let $\boldsymbol{D} = (V, \boldsymbol{B}, \boldsymbol{G})$ be a $GD_\lambda[H]$ with class list $(G_1, \ldots, G_s)$, and let $w: V \to \mathbb{N}_0$ be a weighting. For each block $B = \{x_1, \ldots, x_h\}$ of $\boldsymbol{D}$, assume the existence of a $GD_\mu[K]$, say $\boldsymbol{D}_B$, with class type $(w(x_1), \ldots, w(x_h))$. Then there exists a $GD_{\lambda\mu}[K]$, say $\boldsymbol{E}$, with class type*

(3.2.a) $\qquad \left(\sum_{x \in G_1} w(x), \ldots, \sum_{x \in G_s} w(x) \right).$

Note: Wilson calls $\boldsymbol{D}$ the *recipe* and the $\boldsymbol{D}_B$ the *ingredients* (for the construction).

Proof. Let the incidence matrix of $\boldsymbol{D}_B$ be

(3.2.b) $\qquad M_B = \begin{array}{|c|}\hline M(x_1, B) \\ \hline M(x_2, B) \\ \hline \ldots\ldots\ldots \\ \hline M(x_h, B) \\ \hline \end{array}$

where each $M(x_i, B)$ is a $w(x_i) \times b_B$-matrix such that $M(x_i, B)M(x_i, B)^T$ is diagonal and $M(x_i, B)M(x_j, B)^T = \mu J$ for $i \neq j$. If $w(x_i) = 0$, then the matrix $M(x_i, B)$ is omitted. Now replace each $1 = a_{xB}$ in the incidence matrix $(a_{x,B})$ of $\boldsymbol{D}$ by the auxiliary matrix $M(x, B)$ defined by (3.2.b), and each $0 = a_{xB}$ by a $w(x) \times b_B$-zero-matrix. In this case write

$$M(x, B) = \begin{pmatrix} 0\,0\ldots 0 \\ \cdots\cdots\cdots \\ 0\,0\ldots 0 \end{pmatrix}.$$

Then the matrix

$$\begin{pmatrix} M(x_1, B_1) & \ldots\ldots\ldots\ldots & M(x_1, B_b) \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ M(x_v, B_1) & \ldots\ldots\ldots\ldots & M(x_v, B_b) \end{pmatrix}$$

describes the desired $GD_{\lambda\mu}[K]$. ■

3.3 Remark. We may w.l.o.g. assume that each $M(x, B) \neq 0$ has a 1 in the upper left corner. If $\lambda = \mu = 1$, then $\boldsymbol{E}$ has a subspace which is isomorphic to $\boldsymbol{D}$. (Consider only the upper left corners of all auxiliary matrices.)

3.4 Corollary. *If $\boldsymbol{D}$ is an $S_\lambda(2, H; v)$, then $\boldsymbol{E}$ has class type*

$$(w(x_1), \ldots, w(x_v)).$$ ■

3.5 Corollary. *Let $\boldsymbol{D}$ be a $GD_\lambda[H, G; v]$ (see* I.6.1*) and suppose that $mk \in GD_\mu(K, m)$ for each $k \in H$. Then there is a $GD_{\lambda\mu}[K, mG; mv]$, say $\boldsymbol{E}$. Here $mG := \{mx : x \in G\} \subseteq \mathbb{N}$.*

Proof. Use Theorem 3.2 with $w(x) = m$ for each $x \in V$. Or directly: replace the 1's in each column of the incidence matrix of $\boldsymbol{D}$ by the partial matrices $M_1, \ldots, M_k$ (see Proposition I.6.2) of the incidence matrix of the respective $GD_\mu[K, m; mk]$. ■

3.6 Corollary. *Let $\boldsymbol{D}$ be a $GD_\lambda[K, G; v]$ and suppose that*

(3.6.a) $\quad m \in TD_\mu(l) \quad$ for $l := \max K$.

Then there is a $GD_{\lambda\mu}[k, mG; mv]$.

Proof. Condition (3.6.a) is equivalent to requiring that $km \in GD_\mu(k, m) \subseteq GD_\mu(K, m)$ for each $k \in K$. Now apply Corollary 3.5. ■

This special case of Theorem 3.2 is used very often, especially in case $\mu = 1$. The most important applications of transversal designs are those given in Lemmas 2.1, 2.2, and Corollary 3.6. Note that MacNeish's Theorem I.7.7 is a special case of Corollary 3.6.

Next we shall present Hanani's recursion lemma which proved to be extremely successful for his existence theorems.

3.7 Definition and Notation. For $K \subseteq \mathbb{N}$ and $m, \lambda \in \mathbb{N}$ define $R^m_{K,\lambda}$ to be the set of all $x \in \mathbb{N}$ such that

(3.7.a) $\qquad mx \in GD_\lambda(K, m).$

For $\lambda = 1$ write R^m_K instead of $R^m_{K,1}$, and if $K = \{k\}$ write k for $\{k\}$. Our definition means

(3.7.b) $\qquad mR^m_{K,\lambda} = GD_\lambda(K, m).$

Wilson's notation for R^m_K is $NG(m, K)$. By Lemma 2.5, we get

(3.7.c) $\qquad R^m_{K,\lambda} = R^m_{B(K),\lambda}.$

Let us remark that Hanani (1975) determined the sets $R^m_{k,\lambda}$ completely for $k = 3$, and Brouwer, Schrijver and Hanani (1977) did the same for $k = 4$, cf. §9.

3.8 Examples. (a) $R^1_{K,\lambda} = B(K, \lambda) = S_\lambda(2, K)$, see Notation I.2.19.

(b) $R_k := R^{k-1}_k = \{x \in \mathbb{N} : (k-1)x \in GD(k, k-1)\}$
$= \{x \in \mathbb{N} : (k-1)x + 1 \in B(k)\}$, see (I.6.5.d).
The notation R_k is customary, see e.g. Wilson (1972a). We chose the notation $R^m_{K,\lambda}$ introduced in 3.7 as a straightforward generalisation.

(c) $g \in TD_\lambda(k) \iff k \in R^g_{k,\lambda}$.

In general, our knowledge about the sets $R^m_{K,\lambda}$ is unsatisfactory. By Corollary VII.6.7, the sets $B(K, \lambda)$ and R_K are infinite for $\max K = k > 1$. The following fundamental lemma is implicit in Hanani's work.

3.9 Main Lemma (Hanani's Recursion Lemma). *If $K, G \subseteq \mathbb{N}$ and $m, \lambda, \mu \in \mathbb{N}$, then*

(3.9.a) $\qquad m \cdot GD_\lambda\big(R^m_{K,\mu}, G\big) \subseteq GD_{\lambda\mu}(K, mG).$

Proof. Put $H := R^m_{K,\mu}$. By hypothesis, $mH = GD_\mu(K, m)$. Now apply Corollary 3.5. ■

These theorems and lemmas are fairly general, and hence they need some explanation by examples. The following results are due to Wilson (1972b).

3.10 Corollary. *The sets $R^m_{K,\lambda}$ are closed. In particular the sets $B(K, \lambda)$ and R_k are closed.*

Proof. In the Main Lemma take $G = \{1\}$, $\lambda = 1$. Then

$$m\, GD\big(R^m_{K,\mu}, 1\big) = m B\big(R^m_{K,\mu}\big) \subseteq GD_\mu(K, m).$$

By (3.7.b),

$$B\big(R^m_{K,\mu}\big) \subseteq R^m_{K,\mu}. \ \blacksquare$$

3.11 Lemma. *Let $K \neq \{1\}$ be a non-empty subset of $\mathbb{N}$. Put*

(3.11.a) $\quad L := B(K, \lambda),$

and suppose that the positive integer m has the property

(3.11.b) $\quad x \equiv 1 \pmod m$ $\quad$ *for all $x \in L$.*

Define $Y \subseteq \mathbb{N}_0$ by

(3.11.c) $\quad L = B(K, \lambda) = mY + 1.$

Then, with the convention $0 \in GD_\lambda(X, Y)$ for all $X, Y \in \mathbb{N}$,

(3.11.d) $\quad GD\big(R^m_{K,\lambda}, Y\big) = Y.$

Proof. By the Main Lemma 3.9,

$$m\, GD\big(R^m_{K,\lambda}, Y\big) = GD_\lambda(K, mY) = GD_\lambda(K,\, B(K, \lambda) - 1).$$

With (2.6.b) and (3.11.c) the assertion follows, as trivially $Y \subseteq GD(R^m_{K,\lambda}, Y)$. ■

Note that the case $m = 1$ is already contained in (2.6.b) and (2.5.a).

3.12 Exercise. If q is a prime power, then

(3.12.a) $\quad q \in R_q^q$,

(3.12.b) $\quad q+1 \in R_q \cap R_{q+1} = R_q^{q-1} \cap R_{q+1}^q$.

3.13 Lemma. *If $m, r \in R_k$ and $m \in TD(k)$, then $mr \in R_k$.*

Proof. By hypothesis, there is a $GD[k, k-1; r(k-1)]$. By Corollary 3.6, we obtain $mr(k-1) \in GD[k, m(k-1), mr(k-1)]$. Hence $mr(k-1)+1 \in B(\{k, m(k-1)+1\}) = B(k)$, which implies the assertion. ■

3.14 Example. Let $\boldsymbol{D}_1$ be an $S(2, 6; 31)$. Since $5 \in TD(6)$, Corollary 3.6 implies the existence of a $GD[6, 5; 155]$, i.e. of an $S(2, 6; 156)$ containing a subspace of order 31. Hence there is a $GD(6)$ with 156 points and class type $125 \cdot (1) + (31)$. Furthermore, there is a $TD[6; 31]$, hence a $GD(6)$ with 186 points and class type $124 \cdot (1) + 2 \cdot (31)$. Moreover, $126 \in B(6)$, by VIII.9.5. Now let $\boldsymbol{D}$ be the projective plane $PG(2, 125)$, and let N be a subset of an oval, $0 \leq |N| \leq 126$. Weight the point set V of $\boldsymbol{D}$ as follows.

$$w(x) = \begin{cases} 31 & \text{if } x \in N \\ 1 & \text{otherwise.} \end{cases}$$

If B is a block of $\boldsymbol{D}$, then $|B \cap N| \leq 2$. If

$$B \cap N = \begin{cases} 0 \\ 1 \\ 2 \end{cases},$$

there is a $GD(6)$ with class type

$$\begin{cases} 126 \cdot (1), \\ 125 \cdot (1) + (31), \\ 124 \cdot (1) + 2 \cdot (31). \end{cases}$$

Hence the hypothesis of Theorem 3.2 is satisfied, and Corollary 3.4 yields $15751 + 30 \cdot \mathbb{N}_0^{126} \subseteq GD(6, \{1, 31\}) \subseteq B(6)$; i.e.

(3.14.a) $\quad [30\,\mathbb{N} + 1]_{15751}^{19531} \subseteq B(6)$.

3.15 Notation. Let $F_K(u)$ denote the set of positive integers for which there exists an $S(2, K; v)$ with a subspace of order u. For instance

$$\begin{aligned} &F_K(0) = F_K(1) = B(K),\\ &F_k(k) = B(k),\\ &F_K(u) = \emptyset \quad \text{if } u \notin B(K),\\ &GD(K, u),\ GD(K, u-1) + 1 \in F_K(u). \end{aligned}$$

Doyen and Wilson (1973) proved that

(3.15.a) $\quad F_3(u) = \{u\} \cup \{[6\,\mathbb{N} + \{1, 3\}]_{2u+1}^{\infty} \quad \text{for } u \in 6\,\mathbb{N} + \{1, 3\}.$

We shall prove this result in Theorem 11.3. Wilson (1972b) calls the following theorem the "Adjunction Theorem".

3.16 Theorem. *Let $\boldsymbol{D} = (V, \boldsymbol{B}, \boldsymbol{G})$ be a $GD[K]$ and $d \in \mathbb{N}$. If*

(3.16.a) $\quad |G| + d \in F_K(d)$ *for each point class G of $\boldsymbol{D}$,*

then, for each $G \in \boldsymbol{G}$,

(3.16.b) $\quad |V| + d \in F_K(|G| + d) \subseteq B(K).$

Proof. Let U be a set of d new points. If $G \in \boldsymbol{G}$, construct an $S(2, K; |G| + d)$ on $G \cup U$, with block set $\boldsymbol{B}_G$, such that U is the point set of a fixed subspace (independent of G). Note that this is always possible, since any subspace of a *PBD* may be replaced by any other subspace of the same order. Then $\boldsymbol{B} + \sum_{G \in \boldsymbol{G}} \boldsymbol{B}_G$ is the block set of the desired $S(2, K; |V| + d)$. ■

3.17 Examples.

$$\begin{aligned} 88 &= 4 \cdot 21 + 4 \in B(\{4, 25\}) = B(4), \quad \text{cf. Example VII.3.2.(a),}\\ 366 &= 6 \cdot 60 + 6 \in F_6(66) \subseteq B(6), \quad \text{cf. Exercise VIII.1.10,}\\ 85 &= 5 \cdot 16 + 5 \in F_5(21) \subseteq B(5), \quad \text{cf. (I.2.19.b).} \end{aligned}$$

§4. Applications of Hanani's Lemmas

Most of the following examples are due to Hanani and Wilson.