# 1 Introduction to physiological calculation: approximation and units

One purpose of the many calculations in later chapters is to demonstrate, as 'an encouragement to quantitative thinking', that a little simple arithmetic can sometimes give useful insights into physiology. Encouragement in this chapter takes the form of suggestions for minimizing some of the common impediments to calculation. I have mainly in mind the kinds of arithmetical problem that can suggest themselves outside the contexts of pre-planned teaching or data analysis. Some of the ideas are elementary, but they are not all as well known as they should be. Much of the arithmetic in this book has deliberately been made easy enough to do in the head (and the calculations and answers are given at the back of the book anyway). However, it is useful to be able to cut corners in arithmetic when a calculator is not to hand and guidance is first given on how and when to do this. Much of this chapter is about physical units, for these have to be understood, and casual calculation is too easily frustrated when conversion factors are not immediately to hand. It is also true that proper attention to units may sometimes propel one's arithmetical thinking to its correct conclusion. Furthermore, analysis in terms of units can also help in the process of understanding the formulae and equations of physiology, and the need to illustrate this provides a pretext for introducing some of these. The chapter ends with a discussion of ways in which exponents and logarithms come into physiology, but even here there is some attention to the topics of units and of approximate calculation.

## 1.1 Arithmetic – speed, approximation and error

We are all well drilled in accurate calculation and there is no need to discuss that; what some people are resistant to is the notion that accuracy may sometimes take second place to speed or convenience. High accuracy in physiology is often unattainable anyway, through the inadequacies of data. These points do merit some discussion. Too much initial concern for accuracy and rigour should not be a deterrent to calculation, and those people who confuse the precision of their calculators with accuracy are urged to cultivate the skills of approximate ('back-of-envelope') arithmetic. Discussed here are these skills, the tolerances implicit in physiological variability, and at times the necessity of making simplifying assumptions.

On the matter of approximation, one example should suffice. Consider the following calculation:

$$311/330 \times 480 \times 6.3.$$

A rough answer is readily obtained as follows:

$$\begin{aligned}&(\text{nearly } 1) \times (\text{just under } 500) \times (\text{just over } 6)\\&\quad = \text{slightly under } 3000.\end{aligned}$$

The 480 has been rounded up and 6.3 rounded down in a way that should roughly cancel out the resulting errors. As it happens, the error in the whole calculation is only 5%.

When is such imprecision acceptable? Here is something more concrete to be calculated: *In a man of 70 kg a typical mass of muscle is 30 kg: what is that as a percentage?* An answer of 42.86% is arithmetically correct, but absurdly precise, for the mass of muscle is only 'typical', and it cannot easily be measured to that accuracy even with careful dissection. An answer of 43%, even 40%, would seem precise enough.

Note, in this example, that the two masses are given as round numbers, each one being subject both to variation from person to person and to error in measurement. This implies some freedom for one or other of the masses to be changed slightly and it so happens that a choice of 28 kg, instead of 30 kg, for the mass of muscle would make the calculation easier. Many of the calculations in this book have been eased for the reader in just this way.

Rough answers will often do, but major error will not. Often the easiest mistake to make is in the order of magnitude, i.e. the number of noughts or the position of the decimal point. Here again the above method of approximation is useful – as a check on order of magnitude when more accurate arithmetic is also required. Other ways of avoiding major error are discussed in Section 1.3.

Obviously, wrong answers can be obtained if the basis of a calculation is at fault. However, some degree of simplification is often sensible as a first step in the exploration of a problem. Many of the calculations in this book involve simplifying assumptions and the reader would be wise to reflect on their appropriateness; there is sometimes a thin line between what is inaccurate, but helpful in the privacy of one's thoughts, and what is respectable in print. Gross simplification can indeed be helpful. Thus, the notion that the area of body surface available for heat loss is proportionately less in large than in small mammals is sometimes first approached, not without some validity, in terms of spherical, limbless bodies. The word 'model' can be useful in such contexts – as a respectable way of acknowledging or emphasizing departures from reality.

## 1.2 Units

Too often the simplest physiological calculations are hampered by the fact that the various quantities involved are expressed in different systems of units for which interconversion factors are not to hand. One source of information may give pressures in mmHg, and another in $cmH_2O$, Pa ($= N/m^2$) or dyne/cm$^2$ – and it may be that two or three such diverse figures need to be combined in the calculation. Spontaneity and enthusiasm suffer, and errors are more likely.

One might therefore advocate a uniform system both for physiology generally and for this book in particular – most obviously the metric *Système International d'Unité* or SI, with its coherent use of kilograms, metres and seconds. However, even if SI units are universally adopted, the older books and journals with non-SI units will remain as sources of quantitative information (and one medical journal, having tried the exclusive use of SI units, abandoned it). This book favours the units that seem most usual in current textbooks and in hospitals and, in any case, the reader is not required to struggle with conversion factors. Only occasionally is elegance lost, as when, in Section 5.10, the law of Laplace, so neat in SI units, is re-expressed in other terms.

Table 1.1 lists some useful conversion factors, even though they are not much needed for the calculations in the book. Rather, the table is for general reference and 'an encouragement to (other) quantitative thinking'. For the same reason, Appendix A supplies some additional physical, chemical and mathematical quantities that can be useful to physiologists. Few of us would wish to learn all of Table 1.1, but, for reasons explained below, readers with little physics should remember that $1\ N = 1\ kg\ m/s^2$, that $1\ J = 1\ N\ m$ and that $1\ W = 1\ J/s$. The factor for converting between calories and joules may also be worth remembering, although '4.1855' could be regarded as over-precise for

Table 1.1. *Conversion factors for units*

| | | |
|---|---|---|
| *Time* | | |
| 1 day (d) | **86,400 s** | 1440 min |
| *Distance* | | |
| **1 metre (m)** | 39.4 inch | |
| 1 foot | 0.305 **m** | |
| 1 km | 0.621 mile | |
| 1 Ångstrom unit | 0.1 nanometre (nm) | |
| *Volume* | | |
| 1 litre (l) | **$10^{-3}$ m$^3$** | 1 dm$^3$ |
| *Velocity* | | |
| 1 mph | **0.447 m/s** | 1.609 km/h |
| *Acceleration* (gravitational) | | |
| *g* | **9.807 m/s$^2$** | 32.17 ft/s$^2$ |
| *Mass* | | |
| 1 lb | **0.4536 kg** | 16 oz (avoirdupois) |
| *Force* | | |
| **1 newton (N)** | **1 kg m/s$^2$** | 102 g-force |
| 1 kg-force | **9.807 N** | 1 kilopond |
| 1 dyne | **$10^{-5}$ N** | 1 g cm/s$^2$ |
| *Energy* | | |
| **1 joule (J)** | **1 N m** | |
| 1 erg | **$10^{-7}$ N m** | 1 dyne cm |
| 1 calorie (cal) | **4.1855 J** | |
| 1 m kg-force (1 kg m) | **9.807 J** | |
| *Power* | | |
| **1 watt (W)** | **1 J/s** | 860 cal/h |
| *Pressure and stress* | | |
| **1 N/m$^2$** | 1 pascal (Pa) | |
| 1 kg-force/m$^2$ | **9.807 N/m$^2$** | 1 mmH$_2$O |
| 1 torr | 1 mmHg | 13.6 mmH$_2$O |
| 1 mmHg | **133.3 N/m$^2$** | 0.1333 kPa |
| 750 mmHg | **100.0 kN/m$^2$** | |
| 1 atmosphere | **101.3 kN/m$^2$** | 760 mmHg |

*Note:* SI units, fundamental or derived, are in **bold** lettering.

most purposes. In a similar vein, the '9.807' can often be rounded to '10', but it is best written to at least two significant figures (9.8) since, especially without units, its identity is then more apparent than that of commonplace '10'. It helps to have a feeling for the force of 1 N in terms of weight; it is approximately that of a 100-g object – Newton's legendary apple perhaps. As for pressure, 1 kg-force/m² and 9.807 N/m² may be better appreciated as 1 $mmH_2O$, which is perhaps more obviously small.

Units may be written, for example, in the form m/s² or m s⁻². I have chosen what I believe to be the more familiar style. The solidus (/) may be read as 'divided by' or as 'per', and often these meanings are equivalent. However, there is the possibility of ambiguity when more than one solidus is used, and that practice is best avoided. We shortly meet (for solubility coefficients) a combination of units that can be written unambiguously as 'mmol/l per mmHg', 'mmol/l mmHg', 'mmol/(l mmHg)' and 'mmol $l^{-1}$ $mmHg^{-1}$'. What is ambiguous is 'mmol/l/mmHg', for if each solidus is read as 'divided by' rather than as 'per', then the whole combination would be wrongly read as 'mmol mmHg/l'. In the course of calculations, e.g. involving the cancellation of units (see below), it can be helpful to make use of a horizontal line to indicate division, so that 'mmol/l per mmHg' becomes:

$$\frac{\text{mmol/l}}{\text{mmHg}} \text{ or } \frac{\text{mmol}}{\text{l mmHg}}.$$

### 1.3 How attention to units can ease calculations, prevent mistakes and provide a check on formulae

Students often quote quantities without specifying units, thereby usually making the figures meaningless. All know that units and their interconversions have to be correct, but the benefits of keeping track of units when calculating are not always fully appreciated. Thus, their inclusion in all stages of a calculation can prevent mistakes of various kinds. Indeed, attention to units can sometimes lead to correct answers (e.g. when tiredness makes other reasoning falter), or help in checking the correctness of half-remembered formulae. Too many people flounder for lack of these simple notions. The illustrations that follow involve commonplace physiological formulae, but if some of them are unfamiliar that could even help here, by making the usefulness of the approach more apparent. The formulae are in a sense incidental, but, since they are useful in their own right, the associated topics are highlighted in bold type.

To illustrate the approach I start with an example so simple that the benefits of including units in the calculation may not be apparent. It concerns the excretion of urea. An individual is producing urine at an average rate of, say, 65 ml/h. The average concentration of urea in the urine is 0.23 mmol/ml. The rate of urea excretion may be calculated as the product of these quantities, namely 65 ml/h × 0.23 mmol/ml. The individual units (ml, mmol and min) are to be treated as algebraic quantities that can be multiplied, divided or cancelled as appropriate. Therefore, for clarity, the calculation may be written out thus:

$$65\,\frac{\text{ml}}{\text{h}} \times 0.23\,\frac{\text{mmol}}{\text{ml}} = 15\,\frac{\text{mmol}}{\text{h}},\ \text{i.e. } 15\ \text{mmol/h}.$$

With the units spelt out like that, it would immediately become apparent if, say, there were an inappropriate mixing of volume units, e.g. millilitres in 'ml/h' with litres in 'mmol/l'. (What would then need to be done is probably obvious, but there is one particular kind of procedure for introducing conversion factors – in this case the '1000' relating ml to l – that can be helpful when one is trying to calculate with units in an orderly fashion; see Notes and Answers, note 1.3A.) It would also be obvious if the mistake were made of dividing insteading of multiplying – since the 'ml' would not then cancel. If unsure whether to multiply the two quantities together, or to divide one by the other, one would only have to try out the three possible calculations to see which one yields a combination of units appropriate to excretion rate, i.e. mmol/h and not, say, ml$^2$/(mmol h).

The calculation of **rates of substance flow** from products of concentration and fluid flow in that way is commonplace in physiology and the idea leads directly to the concept of **renal clearance**, and specifically to the use of inulin clearance as a measure of glomerular filtration rate (GFR). Often, when I have questioned students about inulin clearance, they have been quick to quote an appropriate formula, but have been unable to suggest appropriate units for what it yields. It is the analysis of the formula in terms of units that is my ultimate concern here, but a few lines on its background and derivation may be appropriate too. For the measurement of GFR, the plant polysaccharide inulin is infused into the body and measurements are later made of the concentrations in the blood plasma ($P$) and urine ($U$) and of the rate of urine flow ($V$). The method depends on two facts: first, that the concentration in the glomerular filtrate is essentially the same as the concentration in the plasma and, second, that the amount of inulin excreted is equal to the amount filtered. The rate of excretion is $UV$ (as for urea) and the rate of filtration is GFR $\times$ $P$ (again a flow times a concentration). Thus:

$$\text{GFR} \times P = UV,$$

so that:

$$\text{GFR} = \frac{UV}{P}. \tag{1.1}$$

Although the quantity calculated here is the GFR, it can also be thought of as the rate at which plasma would need to be completely cleared of inulin to explain the excretion rate (whereas in fact a larger volume is partially cleared). Hence the term 'renal plasma clearance'. The formula may be generalized to calculate clearances for other excreted substances:

$$\text{renal plasma clearance} = \frac{UV}{P}. \tag{1.2}$$

It may be obvious that GFR needs to be expressed in terms of a volume per unit time, but for the more abstruse concept of clearance the appropriate units are less apparent. This brings us to my main point, that appropriate units can be found by analysis of the formula.

If the concentrations are expressed as g/ml, and the urine flow rate is expressed as ml/min, then the equation can be written in terms of these units as follows:

$$\text{units for clearance} = \frac{\text{g/ml} \times \text{ml/min}}{\text{g/ml}}.$$

Since 'g/ml' appears on the top and bottom lines, it can be cancelled, leaving the right-hand side of the equation as 'ml/min'. Such units (volume per unit time) are as appropriate to clearances in general as to GFR.

To reinforce points made earlier, suppose now that equation 1.1 is wrongly remembered, or that the concentrations of inulin in the two fluids are expressed differently, say one as g/l and one as g/ml. If the calculation is written out with units, as advocated, then error is averted.

It has been emphasized that rates of substance flow can be calculated as products of concentration and fluid flow. In another context, the rate of oxygen flow in blood may be calculated as the product of blood oxygen content and blood flow, and the rate of carbon dioxide loss from the body may be calculated as the product of the concentration (or percentage) of the gas in expired air and the respiratory minute volume. Such ideas lead straight to the **Fick Principle** as applied, for example, to the estimation of cardiac output from measurements of whole-body oxygen consumption and concentrations of oxygen in arterial and mixed-venous blood. The assumption is that the oxygen consumption is equal to the difference between the rates at which oxygen flows to, and away from, the tissues:

$$\begin{aligned}&\text{oxygen consumption}\\&= \text{cardiac output} \times \text{ arterial } [O_2] - \text{cardiac output} \times \text{mixed-venous } [O_2]\\&= \text{cardiac output} \times (\text{arterial } [O_2] - \text{mixed-venous } [O_2]),\end{aligned}$$

where the square brackets indicate concentrations. From this is derived the Fick Principle formula:

$$\text{cardiac output} = \frac{\text{oxygen consumption}}{\text{arterial } [O_2] - \text{mixed-venous } [O_2]}. \qquad (1.3)$$

Re-expressed in terms of units, this becomes:

$$\text{cardiac output} = \frac{\text{ml } O_2/\text{min}}{\text{ml } O_2/\text{l blood}} = \frac{\text{ml } O_2}{\text{min}} \times \frac{\text{l blood}}{\text{ml } O_2} = \text{l blood/min}.$$

Note two points. First, mistakes may be avoided if the substances (oxygen and blood) are specified in association with the units ('ml $O_2$/l blood' rather than 'ml/l'). Second, the two items in the bottom line of equation 1.3 have the same units and are lumped together in the treatment of units. Actually, since one is subtracted from the other, it is a necessity that they share the same units. Indeed, if one finds oneself trying to add or subtract quantities with different units, then one should be forced to recognize that the calculation is going astray.

We turn now to the **mechanical work** that is done when an object is lifted and when blood is pumped. When a force acts over a distance, the mechanical work done is equal to the product of force and distance. Force may be expressed in newtons and distance in metres. Therefore, work may be expressed in N m, the product of the two, but also in joules, since 1 J = 1 N m (Table 1.1). Conversion to calories, etc. is also possible, but the main point here is something else. When an object is lifted, the work is done against gravity, the force being equal (and opposite) to the object's weight. Weights are commonly expressed as 'g' or 'kg', but these are actually measures of mass and not of force, whereas the word 'weight' should strictly be used for the downward force produced by gravity acting on mass. A mass of 1 kg may be

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more properly spoken of as having a weight of 1 kg-force. Weight depends on the strength of gravity, the latter being expressed in terms of *g*, the gravitational acceleration. This is less on the Moon than here, and it is variable on the Earth in the third significant figure, but for the purpose of defining 'kg-force' the value used is 9.807 m/s$^2$, with 1 kg-force being 9.807 N (Table 1.1). This distinction between mass and weight is essential to the procedures advocated here for analysing equations in terms of units and including units in calculations to avoid error.

In relation to the pumping of blood, the required relationship is not 'work equals force times distance', but 'work equals increase in pressure times volume pumped'. If unsure of the latter relationship, can one check that it makes sense in terms of units? The analysis needs to be in terms of SI units, not, say, calories, mmHg and litres. Areas are expressed as m$^2$, and volumes as m$^3$. Accordingly:

$$\text{work (J)} = \text{pressure} \times \text{volume} = \text{N/m}^2 \times \text{m}^3 = \frac{\text{N}}{\text{m}^2} \times \text{m}^3 = \text{N m} = \text{J}.$$

Next we have a situation requiring the definition of the newton as 1 kg m/s$^2$. The **pressure due to a head of fluid**, e.g. in blood at the bottom of a vertical blood vessel, is calculated as $\rho gh$, where $\rho$ is the density of the fluid, $g$ is the gravitational acceleration (9.807 m/s$^2$) and $h$ is the height of fluid. To check that this expression really yields units of pressure (N/m$^2$), we write:

$$\rho gh = \frac{\text{kg}}{\text{m}^3} \times \frac{\text{m}}{\text{s}^2} \times \text{m} = \frac{\text{kg}}{\text{m s}^2}$$

Recalling that 1 N = 1 kg m/s$^2$, we now write:

$$\text{pressure} = \frac{\text{N}}{\text{m}^2} = \frac{\text{kg m}}{\text{s}^2} \times \frac{1}{\text{m}^2} = \frac{\text{kg}}{\text{m s}^2},$$

which is the same expression as before.

There are some quantities for which the units are not particularly memorable for most of us, including peripheral resistance and the solubility coefficients for gases in liquids. Appropriate units may be found by analysis of the equations in which they occur. Peripheral resistance is discussed in Section 4.3, while here we consider the case of gas **solubility coefficients**, and specifically the solubility coefficient of oxygen in body fluids such as blood plasma. The concentration of oxygen in simple solution, [$O_2$], increases with the partial pressure, $P_{O_2}$, and with the solubility coefficient, $S_{O_2}$:

$$[O_2] = S_{O_2} P_{O_2}. \tag{1.4}$$

The concentration may be wanted in ml $O_2$/l fluid or in mmol/l, with the partial pressure being specified in mmHg, kPa or atmospheres, but let us choose mmol/l and mmHg. Rearranging equation 1.4 we see that $S_{O_2}$ equals the ratio $[O_2]/P_{O_2}$, so that the compatible solubility coefficient is found by writing:

$$\frac{[O_2]}{P_{O_2}} = \frac{\text{mmol}}{\text{l}} \times \frac{1}{\text{mmHg}} = \frac{\text{mmol/l}}{\text{mmHg}} = \text{mmol/l per mmHg or mmol/l mmHg.}$$

To reinforce the theme of how to avoid errors, note what happens if an incompatible form of solubility coefficient is used in a calculation. In different reference works, solubility coefficients may be found in such forms as 'ml/l per atmosphere', 'mmol/(l Pa)', etc., as well as mmol/l per mmHg. If the first of these versions were to be used in a calculation together with a gas pressure expressed in mmHg, then the units of concentration would work out as:

$$\frac{\text{ml } O_2\text{/l fluid}}{\text{atmosphere}} \times \text{mmHg} = \text{ml } O_2 \text{ mmHg/(l fluid atmosphere).}$$

The need to think again would at once be apparent.

The above illustrations have variously involved SI and non-SI units in accordance with need and convenience, but other methods of analysis are sometimes appropriate that are less specific about units, at least in the early stages. It is mainly to avoid complicating this chapter that a description of 'dimensional analysis' is consigned to Notes and Answers, note 1.3B, but it is also less generally useful than unit analysis. We look next at diffusion to illustrate a slightly different approach in which the choice of units is deferred.

Suppose that an (uncharged) substance S diffuses from region 1 to region 2 along a diffusion distance $d$ and through a cross-sectional area $a$. The (uniform) concentrations of S in the two regions are respectively $[S]_1$ and $[S]_2$. The **rate of diffusion** is given by the following equation:

$$\text{rate} = ([S]_1 - [S]_2) \times a/d \times D, \tag{1.5}$$

where $D$ is the 'diffusion coefficient'. The appropriate units for $D$ may be found by rearranging the equation and proceeding as follows:

$$D = \frac{\text{rate}}{[S]_1 - [S]_2} \times \frac{d}{a} = \frac{\text{rate}}{\text{concentrations}} \times \frac{\text{distance}}{\text{area}}.$$