An Introduction to Radiative Transfer

Methods and applications in astrophysics

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Contents

Chapter 1

Definitions of fundamental quantities of the radiation field

1.1 Specific intensity

This is the most fundamental quantity of the radiation field. We shall be dealing with this quantity throughout this book.

Let dE_ν be the amount of radiant energy in the frequency interval $(\nu, \nu + d\nu)$ transported across an element of area ds and in the element of solid angle $d\omega$ during the time interval dt. This energy is given by

$$dE_\nu = I_\nu \cos\theta \, d\nu \, d\sigma \, d\omega \, dt, \tag{1.1.1}$$

where θ is the angle that the beam of radiation makes with the outward normal to the area ds, and I_ν is the *specific intensity* or simply *intensity* (see figure 1.1).

The dimensions of the intensity are, in CGS units, erg cm^{-2} s^{-1} hz^{-1} ster^{-1}. The intensity changes in space, direction, time and frequency in a medium that absorbs

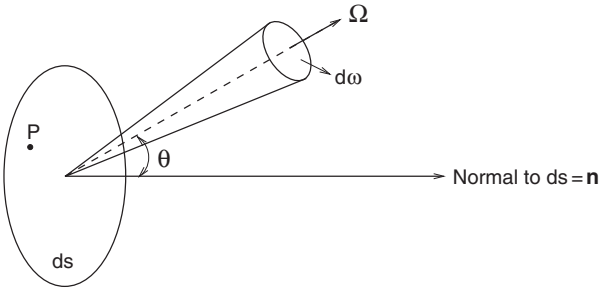

Figure 1.1 Schematic diagram which shows how the specific intensity is defined.

and emits radiation. I_ν can be written as

$$I_\nu = I_\nu(\mathbf{r}, \Omega, t), \tag{1.1.2}$$

where $\mathbf{r}$ is the position vector and Ω is the direction. In Cartesian coordinates it can be written as

$$I_\nu = I_\nu(x, y, z; \alpha, \beta, \gamma; t), \tag{1.1.3}$$

where x, y, z are the Cartesian coordinate axes and α, β, γ are the direction cosines. If the medium is stratified in plane parallel layers, then

$$I_\nu = I_\nu(z, \theta, \varphi; t), \tag{1.1.4}$$

where z is the height in the direction normal to the plane of stratification and θ and φ are the polar and azimuthal angles respectively. If I_ν is independent of φ, then we have a radiation field with axial symmetry about the z-axis. Instead of z, we may choose symmetry around the x-axis.

In spherical symmetry, I_ν is

$$I_\nu = I_\nu(r, \theta; t), \tag{1.1.5}$$

where r is the radius of the sphere and θ is the angle made by the direction of the ray with the radius vector.

The radiation field is said to be isotropic at a point, if the intensity is independent of direction at that point and then

$$I_\nu = I_\nu(\mathbf{r}, t). \tag{1.1.6}$$

If the intensity is independent of the spatial coordinates and direction, the radiation field is said to be homogeneous and isotropic. If the intensity I_ν is integrated over all the frequencies, it is called the integrated intensity I and is given by

$$I = \int_0^\infty I_\nu \, d\nu. \tag{1.1.7}$$

There are other parameters that characterize the state of polarization in a radiation field. These are studied in chapters 11 and 12.

1.2 Net flux

The flux F_ν is the amount of radiant energy transferred across a unit area in unit time in unit frequency interval. The amount of radiant energy in the area ds in the direction θ (see figure 1.1) to the normal, in the solid angle $d\omega$, in time dt and in

the frequency interval $(\nu, \nu + d\nu)$ is equal to $I_\nu \cos\theta \, d\omega \, d\nu \, ds \, dt$. The net flow in all directions is

$$d\nu \, ds \, dt \int I_\nu \cos\theta \, d\omega,$$

or

$$F_\nu = \int I_\nu \cos\theta \, d\omega. \tag{1.2.1}$$

The integration is over all solid angles. This is the net flux and is the rate of flow of radiant energy per unit area per unit frequency.

In polar coordinates, where the outward normal is in the z-direction, we have

$$d\omega = \sin\theta \, d\theta \, d\varphi, \tag{1.2.2}$$

where φ is the azimuthal angle. The net flux F_ν then becomes

$$F_\nu = \int_0^{2\pi} \int_0^{\pi} I_\nu \cos\theta \sin\theta \, d\varphi \, d\theta. \tag{1.2.3}$$

The dimensions of flux are erg cm^{-2} s^{-1} hz^{-1}. Equation (1.2.3) can also be written as

$$\begin{aligned} F_\nu &= \int_0^{2\pi} d\varphi \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta + \int_0^{2\pi} d\varphi \int_{\pi/2}^{\pi} I_\nu \cos\theta \sin\theta \, d\theta \\ &= F_\nu(+) - F_\nu(-), \end{aligned} \tag{1.2.4}$$

where

$$F_\nu(+) = \int_0^{2\pi} \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta \, d\varphi \tag{1.2.5}$$

and

$$F_\nu(-) = \int_0^{2\pi} \int_{\pi}^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta \, d\varphi. \tag{1.2.6}$$

The physical meaning of equation (1.2.4) is as follows: $F_\nu(+)$ represents the radiation illuminating the area from one side and $F_\nu(-)$ represents the radiation illuminating the area from another side. Therefore F_ν, the flux of radiation transported through the area, is the difference between these illuminations of the area. The flux depends on the direction of the normal to the area. The dependence of the flux on direction shows that flux is of vector character. In the Cartesian coordinate system, let the angles made by the direction of radiation with the axes x, y and z be α_1, β_1 and γ_1 respectively, then the flux or radiation along the coordinate axes is given by

$$F_\nu(x) = \int I_\nu \cos\alpha_1 \, d\omega, \tag{1.2.7}$$

$$F_\nu(y) = \int I_\nu \cos\beta_1 \, d\omega, \tag{1.2.8}$$

$$F_\nu(z) = \int I_\nu \cos\gamma_1 \, d\omega. \tag{1.2.9}$$

Furthermore, if α_2, β_2 and γ_2 are the angles made by the coordinate axes and the normal to the area and θ is the angle between the normal and the direction of the radiation, then

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \tag{1.2.10}$$

Substituting equation (1.2.10) into equation (1.2.1), we get

$$F_\nu = \cos\alpha_2 \, F_\nu(x) + \cos\beta_2 \, F_\nu(y) + \cos\gamma_2 \, F_\nu(z). \tag{1.2.11}$$

The integrated flux over frequency is

$$F = \int_0^\infty F_\nu \, d\nu. \tag{1.2.12}$$

If the radiation field is symmetric with respect to the coordinate axes, then the net flux across the surface oriented perpendicular to that axis is zero as the oppositely directed rays cancel each other. In a homogeneous planar geometry, $F_\nu(x)$ and $F_\nu(y)$ are zeros and only $F_\nu(z)$ exists. In such a situation, we have

$$F_\nu(z,t) = 2\pi \int_{-1}^{+1} I(z,\mu,t)\mu \, d\mu, \tag{1.2.13}$$

where $\mu = \cos\theta$.

The astrophysical flux $F_{A\nu}(z,t)$ normally absorbs the π on the RHS of equation (1.2.13) and is written as

$$F_{A\nu}(z,t) = 2 \int_{-1}^{+1} I(z,\mu,t)\mu \, d\mu \tag{1.2.14}$$

and the Eddington flux $F_{E\nu}$ is defined as

$$F_{E\nu}(z,t) = \frac{1}{2} \int_{-1}^{+1} I(z,\mu,t)\mu \, d\mu. \tag{1.2.15}$$

1.2.1 Specific luminosity

The specific luminosity was suggested by Rybicki (1969) and Kandel (1973). We define it following Collins (1973).

From figure 1.2, we define the specific luminosity $\mathcal{L}(\psi,\xi)$ in terms of the orientation variables ψ and ξ as

$$\mathcal{L}(\psi,\xi) = 4\pi \int_A I(\theta,\phi)\hat{n}(\theta,\phi)\cdot\hat{o}(\theta,\phi) \, dA(\theta,\phi), \tag{1.2.16}$$

where $\hat{n}(\theta,\phi)$ and $\hat{o}(\theta,\phi)$ are position dependent unit vectors normal to the surface and in the direction of the observer respectively. The area A over which the specific

intensity $I(\theta, \phi)$ is to be integrated is the 'observable' surface and is defined by the orientation angles ψ and ξ. It is obvious from equation (1.2.16) that $\mathcal{L}(\psi, \xi)$ is a function of the orientation of the object with respect to the observer and is measured per unit solid angle; the total luminosity L is given in terms of $\mathcal{L}(\psi, \xi)$ as

$$L = \frac{1}{4\pi} \int_{4\pi} \mathcal{L}(\psi, \xi)\, d\Omega(\psi, \xi). \tag{1.2.17}$$

1.3 Density of radiation and mean intensity

Let V and Σ be two regions (see figure 1.3) the latter being larger than the former in linear dimensions but sufficiently small for a pencil not to have its intensity changed appreciably in transit. The radiation travelling through V must have crossed the region Σ through some element; let $d\Sigma$ be such an element with normal **N**. The

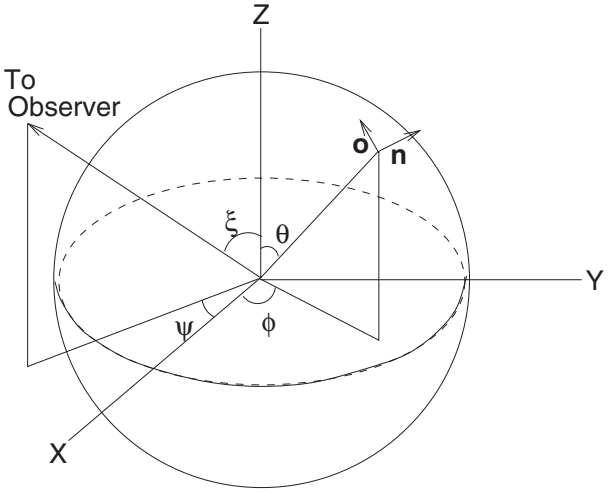

Figure 1.2 The angles θ and ϕ are the angular coordinates of a point on the stellar surface, and therefore represent a local structure. The angles ψ and ξ represent the orientation of the stellar body (from Collins (1973), with permission).

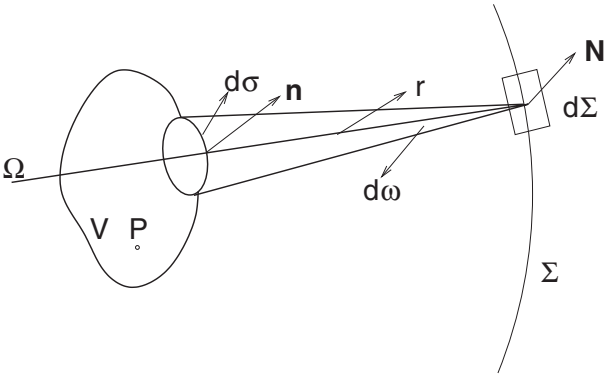

Figure 1.3 Schematic diagram to define density of radiation.

energy passing through $d\Sigma$ which also passes through $d\sigma$ with normal $\mathbf{n}$ on V per unit time is

$$I_\nu(\mathbf{\Omega}, \mathbf{N})\, d\Sigma\, d\omega'\, d\nu, \tag{1.3.1}$$

where

$$d\omega' = (\mathbf{\Omega} \cdot \mathbf{n})\, d\sigma / r^2. \tag{1.3.2}$$

If l is the length travelled by the pencil in V, then an amount of energy

$$\frac{I_\nu(\mathbf{\Omega} \cdot \mathbf{n})(\mathbf{\Omega} \cdot \mathbf{N})\, d\sigma\, d\Sigma\, d\nu}{r^2} \frac{l}{c} \tag{1.3.3}$$

will have travelled through the element in time l/c, where c is the velocity of light.

The solid angle $d\omega$ subtended by $d\Sigma$ at P is $(\Omega \cdot \mathbf{N})\, d\Sigma / r^2$ and the volume intercepted in V by the pencil is given by

$$dV = l(\mathbf{\Omega} \cdot \mathbf{n})\, d\sigma. \tag{1.3.4}$$

This amount of energy is given by

$$\frac{1}{c} I_\nu\, d\nu\, dV\, d\omega. \tag{1.3.5}$$

Therefore, the contribution to the energy per unit volume per unit frequency range (in the interval $\nu, \nu + d\nu$) coming from the solid angle $d\omega$ about the direction Ω is $I_\nu\, d\omega / c$ and the energy density is defined as

$$U_\nu = \frac{1}{c} \int I_\nu\, d\omega. \tag{1.3.6}$$

The average intensity or mean intensity J_ν is

$$J_\nu = \frac{1}{4\pi} \int I_\nu\, d\omega, \tag{1.3.7}$$

so that

$$U_\nu = \frac{4\pi}{c} J_\nu. \tag{1.3.8}$$

For an axially symmetric radiation field, J_ν is given by

$$\begin{aligned} J_\nu &= \frac{1}{2} \int_0^\pi I_\nu \sin\theta\, d\theta \\ &= \frac{1}{2} \int_{-1}^{+1} I(\mu)\, d\mu. \end{aligned} \tag{1.3.9}$$

The integrated energy density U is

$$U = \int_0^\infty U_\nu\, d\nu = \frac{1}{c} \int I\, d\omega. \tag{1.3.10}$$

The dimensions of energy density are erg cm^{-3} hz^{-1} and those of the integrated energy density are erg cm^{-3}. The dimensions of the mean intensity are erg cm^{-2} s^{-1} hz^{-1}.

1.4 Radiation pressure

A quantum of energy $h\nu$ will have a momentum of $h\nu/c$, where c is the velocity of light in the direction of propagation. The pressure of radiation at the point P (see figure 1.1) is calculated from the net rate of transfer of momentum normal to an area ds, which contains the point P. The amount of radiant energy in the frequency range $(\nu, \nu + d\nu)$ incident on ds making an angle θ with the normal to ds traversing the solid angle $d\omega$ in time dt is

$$I_\nu \cos\theta \, d\omega \, d\nu \, ds \, dt. \tag{1.4.1}$$

The momentum associated with this energy in the direction I_ν is

$$\frac{1}{c} I_\nu \cos\theta \, d\omega \, d\nu \, ds \, dt. \tag{1.4.2}$$

Therefore the normal component of the momentum transferred across ds by the radiation is

$$\frac{1}{c} \, d\sigma \, dt \, I_\nu \cos^2\theta \, d\omega \, dt. \tag{1.4.3}$$

The net transfer of momentum across ds by the radiation in the frequency interval $(\nu, \nu + d\nu)$ is

$$\frac{d\sigma \, dt}{c} \int I_\nu \cos^2\theta \, d\omega \, d\nu, \tag{1.4.4}$$

where the integration is over the whole sphere. The pressure at the point P is the net rate of transfer of momentum normal to the element of the surface area containing P in the unit area; the pressure $p_r(\nu) \, d\nu$ can be written in the frequency interval as

$$p_r(\nu) = \frac{1}{c} \int_0^{2\pi} \int_0^{\pi} I_\nu \cos^2\theta \sin\theta \, d\theta \, d\varphi. \tag{1.4.5}$$

If the radiation field is isotropic, then

$$p_r(\nu) = \frac{2\pi}{c} I_\nu \int_0^{\pi} \mu^2 \, d\mu = \frac{4}{3} \frac{\pi}{c} I_\nu \qquad (\mu = \cos\theta) \tag{1.4.6}$$

or in terms of energy density U_ν

$$p_r(\nu) = \frac{1}{3} U_\nu. \tag{1.4.7}$$

The radiation pressure integrated over all frequencies is

$$p_r = \int_0^{\infty} p_r(\nu) \, d\nu \tag{1.4.8}$$

or

$$p_r = \frac{1}{c} \int I \cos^2\theta \, d\omega, \tag{1.4.9}$$

where I is the integrated intensity. Furthermore

$$p_r = \frac{1}{3}U. \tag{1.4.10}$$

It can be seen that the dimensions of radiation pressure are the same as those of energy density, that is, erg cm^{-3} hz $^{-1}$ and the integrated radiation pressure has the dimensions of erg cm^{-3}.

1.5 Moments of the radiation field

Moments are defined in such a way that the nth moment over the radiation field is given by

$$M_n(z, n) = \frac{1}{2}\int_{-1}^{+1} I_\nu(z, \mu)\mu^n \, d\mu. \tag{1.5.1}$$

Following Eddington, we can have the zeroth, first and second moments as:

1. Zeroth moment (mean intensity):

$$J_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I(z, \mu)\, d\mu. \tag{1.5.2}$$

2. First moment (Eddington flux):

$$H_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I(z, \mu)\mu \, d\mu. \tag{1.5.3}$$

3. Second moment (the so called K-integral):

$$K_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I(z, \mu)\mu^2 \, d\mu. \tag{1.5.4}$$

1.6 Pressure tensor

The rate of transfer of the x-component of the momentum across the element of surface normal to the x-direction by radiation in the solid angle dw per unit area in the direction whose direction cosines are l, m, n is

$$\frac{1}{c} Il \, d\omega \, l, \tag{1.6.1}$$

where I is the integrated radiation. If monochromatic radiation is considered, then I should be replaced by $I_\nu \, d\nu$. The total rate of x-momentum transfer across the element per unit area is $p_r(xx)$:

$$p_r(xx) = \frac{1}{c}\int Il^2\, d\omega. \tag{1.6.2}$$

Similarly the y- and z-components are given by

$$p_r(xy) = \frac{1}{c}\int Ilm\, d\omega \quad \text{and} \quad p_r(xz) = \frac{1}{c}\int Iln\, d\omega. \tag{1.6.3}$$

The quantities $p_r(yx)$, $p_r(yy)$, $p_r(yz)$, $p_r(zx)$, $p_r(zy)$ and $p_r(zz)$ are similarly defined for elements of the surfaces normal to the y- and z-directions. These nine quantities constitute the 'stress tensor'.

One can see that $p_r(xy) = p_r(yx)$, $p_r(xz) = p_r(zx)$ and $p_r(yz) = p_r(zy)$ or that the tensor is symmetrical. The mean pressure $\bar{p}$ is defined by

$$\bar{p} = \frac{1}{3}[p_r(xx) + p_r(yy) + p_r(zz)], \tag{1.6.4}$$

and

$$\bar{p} = \frac{1}{3c}\int I\omega = \frac{1}{3}U, \tag{1.6.5}$$

as $l^2 + m^2 + n^2 = 1$.

In the case of an isotropic radiation field

$$\bar{p} = p_r(xx) = p_r(yy) = p_r(zz) = \frac{1}{3}U, \tag{1.6.6}$$

and

$$\left.\begin{aligned} p_r(xy) &= p_r(yx) = 0,\\ p_r(xz) &= p_r(zx) = 0,\\ p_r(yz) &= p_r(xy) = 0. \end{aligned}\right\} \tag{1.6.7}$$

1.7 Extinction coefficient: true absorption and scattering

A pencil of radiation of intensity I_ν is attenuated while passing through matter of thickness ds and its intensity becomes $I_\nu + dI_\nu$, where

$$dI_\nu = -I_\nu \kappa_\nu\, ds. \tag{1.7.1}$$

The quantity κ_ν is called the mass extinction coefficient or the mass absorption coefficient. κ_ν comprises two important processes: (1) true absorption and (2) scattering. Therefore we can write

$$\kappa_\nu = \kappa_\nu^a + \sigma_\nu, \tag{1.7.2}$$

where κ_ν^a and σ_ν are the absorption and scattering coefficients respectively. Absorption is the removal of radiation from the pencil of the beam by a process

which involves changing the internal degrees of freedom of an atom or a molecule. Examples of these processes are: (1) photoionization or bound–free absorption by which the photon is absorbed and the excess energy, if any, goes into the kinetic energy of the electron thermalizing the medium; (2) the absorption of a photon by a freely moving electron that changes its kinetic energy which is known as free–free absorption; (3) the absorption of a photon by an atom leading to excitation from one bound state to another bound state, which is called bound–bound absorption or photoexcitation; (4) the collision of an atom in a photoexcited state which will contribute to the thermal pool; (5) the photoexcitation of an atom which ultimately leads to fluorescence; (6) negative hydrogen absorption, etc. The reversal of the above processes may contribute to the emission coefficient (see section 1.8).

The coefficient κ_ν^a depends on the thermodynamic state of the matter at (pressure p, temperature T, chemical abundances α_i) any given point in the medium. At the point r the coefficient is given by

$$\kappa_\nu^a(r, T) = \kappa_\nu^a\,[p(r, T), T(r), \alpha_i(r, T), \ldots, \alpha_\kappa(r, T)], \tag{1.7.3}$$

when there is local thermodynamic equilibrium (LTE). This kind of situation does not exist in reality and one needs to determine the κ_ν^a in a non-LTE situation. In static media κ_ν^a is isotropic while in moving media it is angle and frequency dependent due to Doppler shifts.

Another process by which energy is lost from the beam is the scattering of radiation which is represented by the mass scattering coefficient κ_ν^s. Scattering changes not only the photon's direction but also its energy. If we define the *albedo for single scattering* as ω_ν, then

$$\omega_\nu = \frac{\sigma_\nu}{\kappa_\nu}, \tag{1.7.4}$$

is the ratio of scattering to the extinction coefficients.

The extinction coefficient is the product of the atomic absorption coefficients or scattering coefficients (cm^2) and the number density of the absorbing or scattering particles (cm^{-3}). The dimension of κ_ν is cm^{-1} and $1/\kappa_\nu$ gives the photon mean free path which is the distance over which a photon travels before it is removed from the pencil of the beam of radiation.

1.8 Emission coefficient

Let an element of mass with a volume element dV emit an amount of energy dE_ν into an element of solid angle $d\omega$ centred around $\mathbf{\Omega}$ in the frequency interval ν to $\nu + d\nu$ and time interval t to $t + dt$. Then

$$dE_\nu = j_\nu\, dV\, d\omega\, d\nu\, dt, \tag{1.8.1}$$

where j_ν is called the macroscopic emission coefficient or emissivity. The emissivity has dimensions erg cm^{-3} sr^{-1} hz^{-1} s^{-1}. Emission is the combination of the reverse of the physical processes that cause true absorption. These processes are: (a) radiative recombination: when a free electron occupies a bound state creating a photon whose energy is the sum of the kinetic energy of the electron and the binding energy; (b) bremsstrahlung: a free electron moving in one hyperbolic orbit emits a photon by moving into a different hyperbolic orbit of lower energy; (c) photo de-excitation or collisional de-excitation: a bound electron changes to another bound state by emitting a photon through collision; (d) collisional recombination: a photoexcited atom contributes photon energy by collisional ionization; the reverse of this is called (three-body) collisional recombination; and (e) fluorescence: if a photon is absorbed by an atom and it is excited from bound state p to another bound state r, decays to an intermediate bound state q and then to the original state p, this process is called fluorescence. The energy from the original absorbed photon is re-emitted in two photons each of different energy.

A true picture of the occupation numbers is obtained only when the statistical equilibrium equation, which describes all necessary processes that are to be taken into account, is written. When LTE exists, the emission coefficient is given by

$$j_\nu^a(LTE) = \kappa_\nu^a B_\nu(T), \tag{1.8.2}$$

where $B_\nu(T)$ is the Planck function:

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}. \tag{1.8.3}$$

Equation (1.8.2) is known as Kirchhoff–Planck relation. In a non-LTE situation one has to consider stimulated emission due to the presence of the radiation field and spontaneous emission and the Einstein transition coefficients involved.

Emission of radiation can also be from the scattered photons. One can write

$$j_\nu^s(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{4\pi}\iint \sigma_\nu^s, (\mathbf{r}, t)\, p(\nu, \mathbf{\Omega}; \nu', \mathbf{\Omega}'; \mathbf{r}, t) I_{\nu'}(\mathbf{r}, \mathbf{\Omega}', t)\, d\nu'\, d\omega'. \tag{1.8.4}$$

The phase function p can be normalized in such a way that

$$\iint p(\nu', \mathbf{\Omega}'; \nu, \Omega, ; \mathbf{r}, t)\, d\nu'\, d\omega' = 4\pi. \tag{1.8.5}$$

This is the manifestation of the conservation of radiation flux, that is, the emitted radiation balances that removed from the beam.

Equation (1.8.2) should be corrected for the stimulated scattering by multiplying it by the correction factor

$$\left\{1 + \frac{c^2}{2h\nu^3} I_\nu(r, \mathbf{\Omega}, t)\right\}. \tag{1.8.6}$$

This makes the transfer equation non-linear in I_ν. Particles, such as ions, atoms, molecules, electrons, solid particles, etc., scatter radiation and contribute to the scattering coefficient.

1.9 The source function

The source function is defined as the ratio of the emission coefficient to the absorption coefficient:

$$S_\nu = j_\nu/\kappa_\nu. \tag{1.9.1}$$

From equations (1.7.4), (1.8.2) and (1.8.4), we can write the source function as

$$S_\nu(\mathbf{r}, \mathbf{\Omega}, t) = [1 - \omega_\nu(r,t)]B_\nu(r,t) + \frac{\omega_\nu(r,t)}{4\pi} \int\int p(\nu', \Omega'; \nu, \mathbf{\Omega}; r, t) I_{\nu'}(\mathbf{r}, \mathbf{\Omega}'; t)\, d\nu'\, d\omega'. \tag{1.9.2}$$

1.10 Local thermodynamic equilibrium

The state of the gas (the distribution of atoms over bound and free states) in thermodynamic equilibrium is uniquely specified by the thermodynamic variables – the absolute temperature T and the total particle density N. The assumption of LTE gives us the freedom to use (in a stellar atmosphere) the local values of T and N in spite of the gradients that exist in the atmosphere. In LTE, the same temperature is used in the velocity distribution of atoms, ions, electrons, etc. Thus the implications of its assumption are drastic. The velocity distribution of the particles is Maxwellian and the degrees of ionization and excitation are determined by the Saha Boltzmann equation (see Mihalas (1978), Sen and Wilson (1998)).

The principle of detailed balance holds good for every transition. This means that the number of radiative transitions $i \to j$ is balanced by the photoexcitation $j \to i$ transitions, where i and j are the upper and lower levels respectively. Thus,

$$n_i \left[A_{ij} + B_{ij} B_{ij}(\nu, T)\right] = n_j B_{ji} B_{ji}(\nu, T) \quad j < i,\ i = 2, \ldots, \tag{1.10.1}$$

where A_{ij}, B_{ij} and B_{ji} are the Einstein coefficients and $B_{ij}(\nu, T)$ and $B_{ji}(\nu, T)$ are the Planck functions given by

$$B_{ij}(\nu, T) = \frac{2h\nu_{ij}^3}{c^2} \left[\exp\left(\frac{h\nu_{ij}}{kT}\right) - 1\right]^{-1}. \tag{1.10.2}$$

The radiative ionization from level i is balanced by radiative recombination to i. This gives us

$$n_e\,[A_{ci} + B_{ci}\,B_{ic}(\nu_{ic}, T)] = n_i\,B_{ic}\,B_{ic}(\nu_{ic}, T), \quad i = 1, 2, \ldots, \tag{1.10.3}$$

for collisional transition, with the detailed balance transitions given by the relations

$$n_i C_{ij} = n_j C_{ji}, \quad i, j = 1, 2, \ldots \quad i \neq j, \tag{1.10.4}$$

where the Cs are collisional rates and the subscript c denotes the continuum.

In the LTE situation, the radiative transitions are negligible compared to collisional transitions. This is an important consideration in treating non-LTE conditions in stellar atmospheres.

1.11 Non-LTE conditions in stellar atmospheres

In LTE conditions the particle distribution is Maxwellian. Every transition is exactly balanced by its inverse transition, that is, the principle of detailed balance holds good in LTE. Generally, the excitation and de-excitation of the atomic levels is caused by radiative and collisional processes. In the interior of the stars collisions dominate over the radiative processes and LTE prevails. Near the surface of the atmosphere, the radiative rates are not in detailed balance and there is a strong departure from the LTE situation and then the non-LTE situation exists and one should adopt a joint detailed balancing of the excitation and de-excitation of atomic levels. The LTE condition can be determined by the comparative contribution of collisional rates and radiative rates – dominance of the former prevails in the LTE situation, while the opposite situation leads to a non-LTE situation. In stellar atmospheres, non-LTE predominates and this should be taken into account in any transfer calculations.

Statistical equilibrium equations describe the equilibrium among various processes leading to the establishment of an equilibrium state. The state of the gas is assumed to be described by its kinetic temperature, the degrees of excitation and the ionization of each atomic level. The equations of statistical equilibrium (or rate equations) are used to calculate the occupation numbers of bound and free states of atoms assuming complete redistribution (that is, the emission and absorption profiles are identical) in a steady atmosphere.

Consider the changes in time of the number of particles in a given state i of a chemical species α in a given volume element of a moving medium. The net rate at which particles are brought to state i by radiative and collisional processes is given by

$$\left(\frac{\partial n_{i\alpha}}{\partial t}\right) = \sum_{j \neq i} n_{j\alpha} P_{ji}^{\alpha} - n_{i\alpha} P_i^{\alpha} + \boldsymbol{\nabla} \cdot (n_{i\alpha} \cdot \mathbf{V}), \tag{1.11.1}$$

where $\mathbf{V}$ is the velocity of the moving medium and P_{ji} represents the total rate of transfer from level j to level i (radiative and collisional). The second term on the RHS gives the total number of particles entering and leaving the volume element,

through the divergence theorem. The total number of particles of type α, N_α, is given by the sum over all states of species α:

$$N_\alpha = \sum_i n_{i\alpha}. \tag{1.11.2}$$

Then we have the continuity equation

$$\left(\frac{\partial N_\alpha}{\partial t}\right) + \boldsymbol{\nabla} \cdot S\,(N_\alpha \mathbf{V}) = 0. \tag{1.11.3}$$

If m_α is the mass of each particle of type α, then by multiplying equation (1.11.3) by m_α and summing over all species of particles in this volume element, we get

$$\rho = \sum_\alpha m_\alpha N_\alpha \tag{1.11.4}$$

and

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{V}) = 0. \tag{1.11.5}$$

If the flow is steady, then

$$\sum_{j \neq i} \left(n_{j\alpha} P^\alpha_{ji} - n_{i\alpha} P^\alpha_{ij}\right) = \boldsymbol{\nabla} \cdot (n_{i\alpha} \mathbf{V}). \tag{1.11.6}$$

If the atmosphere is static, then equation (1.11.6) becomes

$$n_i \sum_{j \neq i} P_{ij} - \sum_{j \neq i} n_j P_{ji} = 0. \tag{1.11.7}$$

We will write a simple model of the statistical equilibrium equation (see Mihalas and Mihalas (1984), pages 386–398 for a detailed account or Mihalas (1978), chapter 5). The equation for the population n_i is

$$\begin{aligned} &\sum_{k=n+1}^{c} n_k (A_{ki} + B_{ki} \bar{J}_{ik} + n_e C_{ki}) + \sum_{j=1}^{i-1} n_j \left(B_{ji} \bar{J}_{ji} + n_e C_{ji}\right) \\ &\quad = n_i \left[\sum_{j=1}^{i-1} \left(A_{ij} + B_{ij} \bar{J}_{ji} + n_e C_{ij}\right) + \sum_{k=i+1}^{c} \left(B_{ik} \bar{J}_{ik} + n_e C_{ik}\right)\right], \end{aligned} \tag{1.11.8}$$

where $\bar{J}$ is the line profile weighted mean intensity. The terms on the LHS of equation (1.11.8) represent different physical quantities: $\sum n_k (A_{ki} + B_{ki} \bar{J}_{ik})$ represents the spontaneous and stimulated radiative transitions from higher discrete levels; $\sum n_k n_e C_{ki}$ represents the collision induced transitions from upper levels; $\sum n_j B_{ji} \bar{J}_{ji}$ represents the photoexcitation from lower levels; and $\sum n_e n_j C_{ji}$ represents the collisional excitation. Similarly the terms on the RHS of (1.11.8) have the following meanings: $n_i \sum (A_{ij} + B_{ij} \bar{J}_{ji})$ represents the spontaneous and stimulated transitions to lower levels; $n_e n_i \sum C_{ij}$ represents the downward transitions induced

by collisions (second kind); $n_i \sum B_{ik} \bar{J}_{ik}$ represents the photoexcitation into higher levels; and $n_e n_i \sum C_{ik}$ represents the upward transitions due to collisions with electrons.

Equation (1.11.8) specifies the gas at a given point in the medium if the radiation field (through $\bar{J}$), temperature and electron density n_e are specified.

1.12 Line source function for a two-level atom

This is one of the most useful quantities in the study of line transfer and has been studied extensively.

Consider two levels 1 and 2 (lower and upper respectively) of an atom. The principle of detailed balance gives us (see Mihalas and Mihalas (1984))

$$g_2 B_{21} = g_1 B_{12} \tag{1.12.1}$$

and

$$A_{21} = \frac{2h\nu_{12}^3}{c^2} B_{21}, \tag{1.12.2}$$

where g_1 and g_2 are the statistical weights, $h\nu_{12}$ is the energy difference between levels 1 and 2 measured relative to the ground state and A and B are the Einstein coefficients. The line absorption coefficient in terms of a convenient width Δs is

$$\kappa_l(\nu) = \frac{h\nu_0}{4\pi\,\Delta s}\,(N_1 B_{12} - N_2 B_{21}), \tag{1.12.3}$$

where N_1 and N_2 are the population densities of levels 1 and 2 respectively and ν_0 is the central frequency of the line. The line source function S_L (see Grant and Peraiah (1972)) is now written as

$$S_L = \frac{A_{21} N_2}{(B_{12} N_1 - B_{21} N_2)}. \tag{1.12.4}$$

We will use the following statistical equilibrium equation for a two-level atom:

$$N_1 \left[B_{12} \int_{-\infty}^{+\infty} \phi(x) J(x)\, dx + C_{12} \right] = N_2 \left[A_{21} + C_{21} + B_{21} \int_{-\infty}^{+\infty} \phi(x) J(x)\, dx \right], \tag{1.12.5}$$

where

$$x = \frac{(\nu - \nu_0)}{\Delta s} \tag{1.12.6}$$

and $\phi(x)$ is the line profile function (see below) and then combining (1.12.4) and (1.12.5) we obtain

$$S_L = (1-\epsilon)\int_{-\infty}^{+\infty} \phi(x) J(x)\, dx + \epsilon B, \tag{1.12.7}$$

where

$$\epsilon = \frac{C_{21}}{C_{21} + A_{21}\left[1 - \exp(h\nu_0/kT)\right]^{-1}} \tag{1.12.8}$$

is the probability per scatter that a photon will be destroyed by collisional de-excitation. When $\epsilon = 1$, LTE prevails and if $\epsilon \ll 1$, a non-LTE situation occurs. In equations (1.12.7) and (1.12.8), B is the Planck function, k is the Boltzmann constant and T is the temperature. Sometimes the line source function is written as

$$S_L = \frac{\bar{J} + \epsilon' B}{1 + \epsilon'}, \tag{1.12.9}$$

where

$$\epsilon' = \epsilon/(1-\epsilon) \tag{1.12.10}$$

and

$$\bar{J} = \int_{-\infty}^{+\infty} \phi(x) J(x)\, dx. \tag{1.12.11}$$

The line profiles are given by (Mihalas 1978):

$$\text{Doppler:} \quad \phi(x) = \pi^{-\frac{1}{2}} \exp(-x^2), \tag{1.12.12}$$

$$\text{Lorentz:} \quad \phi(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \tag{1.12.13}$$

$$\text{Voigt:} \quad \phi(x) = a\pi^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \exp(-x^2)\left[(x-y)^2 + a^2\right] dy, \tag{1.12.14}$$

where a is the ratio of the damping width to the Doppler width ($\Gamma/4\pi\,\Delta\nu_D$).
The profile $\phi(x)$ is normalized such that

$$\int_{-\infty}^{+\infty} \phi(x)\, dx = 1. \tag{1.12.15}$$

1.13 Redistribution functions

In the process of the formation of spectral lines, we assume that scattering is either coherent or completely redistributed over the profile of the line. These assumptions are ideal and not achieved in real stellar atmospheres. It is necessary to find out how after scattering the photons are redistributed in angle and frequency across the line profile. These calculations are described in the form of partial redistribution functions. First, we consider an atom in its own frame of reference and find the

redistribution that happens within the substructure of the bound states. We need to take into account the Doppler redistribution in the frequency produced by the atom's motion. Generally, the directions of the incident and emergent photons are different, therefore the projection of the atom's velocity vector along the propagation vectors will be different for the two photons and a different Doppler shift occurs. This gives rise to the Doppler redistribution. One needs to average over all possible velocities to obtain the final redistribution function. This redistribution function will be used in the line transfer calculation to obtain the correlation (if any) between the incoming and outgoing photons. In what follows, we will give the redistribution functions that will be useful in line transfer (see Hummer (1962), Mihalas (1978)).

The probability of emission of a photon after absorption is

$$R(\nu, \mathbf{q}, \nu', \mathbf{q}')\, d\nu'\, d\Omega'\, d\nu\, d\Omega, \tag{1.13.1}$$

where ν and $\mathbf{q}$ are the frequency and direction of the absorbed photon and ν' and $\mathbf{q}'$ are the frequency and direction of the emitted photon. This probability is subject to the condition

$$\int\int\int\int R(\nu, \mathbf{q}; \nu', \mathbf{q}')\, d\nu'\, d\Omega'\, d\nu\, d\Omega = 1. \tag{1.13.2}$$

Here $d\Omega$ and $d\Omega'$ are the real elements normal to directions $\mathbf{q}$ and $\mathbf{q}'$ respectively. If $\phi(\nu')\, d\nu'$ is the probability that a photon with a frequency in the interval $(\nu, \nu + d\nu)$ is emitted in the interval $(\nu', \nu' + d\nu')$, then

$$4\pi \int\int R(\nu', \mathbf{q}'; \nu, \mathbf{q})\, d\nu\, d\Omega = \phi(\nu', \mathbf{q}'), \tag{1.13.3}$$

where $\phi(\nu', \mathbf{q}')$ is the profile function, which is again subjected to the normalization condition that

$$\int\int \phi(\nu'\mathbf{q}')\, d\nu'\, d\Omega' = 4\pi. \tag{1.13.4}$$

The redistribution functions are given as follows (the roman subscripts are due to Hummer (1962)):

(a) If we have two perfectly sharp upper and lower states in a bound–bound transition, the photons follow a Doppler redistribution. This does not apply to any real line. This redistribution function is given by (see Hummer (1962) and Mihalas (1978))

$$R_{I-AD}(x, \mathbf{q}; x', \mathbf{q}) = \frac{g(\mathbf{q}, \mathbf{q}')}{4\pi^2 \sin\gamma} \exp\left[-x'^2 - \left(x - x' \cos\gamma\right)^2 \operatorname{cosec}^2\gamma\right], \tag{1.13.5}$$

where R_{I-AD} is the angle dependent redistribution function, the x's are the normalized frequencies (see equation (1.12.6)) and γ is the angle between the vectors $\mathbf{q}$ and $\mathbf{q}'$. For isotropic scattering, the phase function is

$$g_{iso}(\mathbf{q}, \mathbf{q}') = \frac{1}{4\pi}, \tag{1.13.6}$$

and for dipole scattering

$$g_{dip}(\mathbf{q}, \mathbf{q}') = \frac{3}{16\pi}\left(1 + \cos^2 \gamma\right). \tag{1.13.7}$$

The redistribution function for isotropic scattering was first obtained by Thomas (1947).

The angle-averaged redistribution function R_{I-A} is given by

$$R_{I-A}(x, x') = \frac{1}{2} \operatorname{erfc} |\bar{x}|, \tag{1.13.8}$$

where

$$\operatorname{erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty \exp(-t^2)\, dt \tag{1.13.9}$$

and

$$|\bar{x}| = \max\left(x, x'\right). \tag{1.13.10}$$

(b) In this case, we have an atom with a perfectly sharp lower state and an upper state broadened by radiative decay or an upper state whose finite life time against radiative decay (back to the lower state) leads to a Lorentz profile. This applies to resonance lines in media of low densities in which collisional broadening of the upper state is negligible, for example, the Lyman alpha line of hydrogen in the interstellar medium. The angle dependent redistribution function is given by

$$R_{II-AD}(x, \mathbf{q}; x', \mathbf{q}') = \frac{g(\mathbf{q}, \mathbf{q}')}{4\pi^2 \sin\gamma} \exp\left[-\left(\frac{x - x'}{2}\right)^2 \operatorname{cosec}^2\left(\frac{\gamma}{2}\right)\right] \times H\left(\sigma \sec\frac{\gamma}{2}, \frac{x + x'}{2} \sec\frac{\gamma}{2}\right), \tag{1.13.11}$$

where $\sigma = \delta/\Delta$, $4\pi\delta$ being the sum of the transition probabilities from the concerned states and Δ the Doppler width given by

$$\Delta = \nu_0 \left(\frac{v}{c}\right), \quad v = \left(\frac{2kT}{m}\right)^{\frac{1}{2}}, \tag{1.13.12}$$

and H is the Voigt function given by

$$H(a, u) = \frac{a}{\pi} \int_{-\infty}^{+\infty} \exp(-y^2) \left[(u - y)^2 + a^2\right]^{-1} dy. \tag{1.13.13}$$

The function R_{II} was first introduced by Henyey (1941).

The angle-averaged R_{II} function is given by

$$R_{II-A}(x, x') = \pi^{-\frac{3}{2}} \int_{\frac{1}{2}|\bar{x}-\underline{x}|}^{\infty} \exp(-u^2) \left[\tan^{-1} \frac{\underline{x}+u}{\sigma} - \tan^{-1} \frac{\bar{x}-u}{\sigma} \right] du, \tag{1.13.14}$$

where $\bar{x} = \max(|x|, |x|')$ and $\underline{x} = \min(|x|, |x|')$. R_{II-A} was first obtained by Unno (1952) and later by Sobolev (1955). Furthermore,

$$\phi(x) = \int_{-\infty}^{+\infty} R_{II-A(iso)}(x, x')\, dx' = H(a, x), \tag{1.13.15}$$

a being the damping constant.

(c) The atom has a perfectly sharp lower state and a collisionally broadened upper state. All the excited electrons are randomly distributed over the substates of the upper states before emission occurs. In this case, the absorption profile is Lorentzian. The damping comprises radiative and collisional rates and represents the full width of the upper state. The redistribution function R_{III} is given by

$$R_{III-AD}(\nu', \mathbf{q}'; \nu, \mathbf{q}) = \frac{g(\mathbf{q}', \mathbf{q})}{\pi^2 \sin\gamma} a \times \int_{-\infty}^{+\infty} \frac{\exp(-u^2) H(a \operatorname{cosec}\gamma, (x - u\cos\theta)\operatorname{cosec}\theta)}{(x-u)^2 + a^2}\, du, \tag{1.13.16}$$

where a is the damping constant of the upper level. Heinzel (1981) gives an R_{III} in laboratory frame which is different from that of Hummer (1962):

$$R_{III-AD}(\nu', \mathbf{q}'; x, \mathbf{q}) = \frac{g(\mathbf{q}', \mathbf{q})}{4\pi^2 \sin\gamma} \left[H\left(a_j \operatorname{cosec}\frac{\gamma}{2}, \frac{x-x'}{2} \operatorname{cosec}\frac{\gamma}{2}\right) \times \exp\left(-\frac{x+x'}{2} \sec^2\frac{\theta}{2}\right) + E_{III}(x', x, \gamma) \right]; \tag{1.13.17}$$

see Heinzel (1981) for $E_{III}(x', x, \gamma)$.

The angle-averaged R_{III-A} is given by

$$R_{III-A}(x', x) = \pi^{-\frac{5}{2}} \int_0^{\infty} \exp(-u^2) \left[\tan^{-1}\left(\frac{x'+u}{a}\right) - \tan^{-1}\left(\frac{x'-u}{a}\right) \right] \times \left[\tan^{-1}\left(\frac{x+u}{a}\right) - \tan^{-1}\left(\frac{x-u}{a}\right) \right] du. \tag{1.13.18}$$

(d) This function applies when a line is formed by an absorption from a broadened state i to a broadened upper state j, followed by a radiative decay to state i. It applies

to scattering in subordinate lines. This was derived by several authors with some controversy but we will quote from Hummer (1962):

$$R_{IV-AD}(x', \mathbf{q}'; x, \mathbf{q}) = \frac{g(\mathbf{q}', \mathbf{q})}{2\pi^2 \sin\gamma} \frac{a_i \sec\frac{\gamma}{2}}{\pi} \times \int_{-\infty}^{+\infty} \frac{\exp(-y^2) H\left(a_j \operatorname{cosec}\frac{\gamma}{2}, y\cot\frac{\gamma}{2} - x\operatorname{cosec}\frac{\gamma}{2}\right)}{\left[(x-x')\sec\frac{\gamma}{2} - 2y\right]^2 + \left(a_i \sec\frac{\gamma}{2}\right)^2} dy, \quad (1.13.19)$$

and the angle-averaged R_{IV} is

$$R_{IV-A}(x', x) = \pi^{-\frac{5}{2}} a_j \int_0^{+\infty} \exp(-u^2)\, du \times \int_{-1}^{+1} \left[\tan^{-1}\left(\frac{x' - x + u(1-\mu)}{a_i}\right) - \tan^{-1}\left(\frac{(x' - x - u(1-\mu)}{a_i}\right)\right] \times \frac{d\mu}{(x-\mu u)^2 + a_j^2} du, \quad (1.13.20)$$

where

$$\mathbf{q} \cdot \mathbf{u} = \mu. \quad (1.13.21)$$

(e) Heinzel (1981) has given R_V, which becomes R_I, R_{II} and R_{III} in special cases. R_V is given in the laboratory reference frame by

$$R_V(x', \mathbf{q}'; x, \mathbf{q}) = \frac{g(\mathbf{q}', \mathbf{q})}{4\pi^2 \sin\gamma} \left[H\left(a_j \sec\frac{\gamma}{2}, \frac{x + x'}{2}\sec\frac{\gamma}{2}\right)\right] \times H\left(a_i \operatorname{cosec}\frac{\gamma}{2}, \frac{x - x'}{2}\operatorname{cosec}\frac{\gamma}{2}\right) + E_V(x', x, \gamma), \quad (1.13.22)$$

where

$$E_V(x', x, \gamma) = \frac{4}{\pi} \int_{v=0}^{\infty} \int_{u=\epsilon v}^{\infty} \exp\left[-u^2 - v^2 - 2A_j u\right] \times \left[\exp(-2A_j u) - \exp(-2A_i \epsilon v)\right] \cos Cu \cos Du\, du\, dv, \quad (1.13.23)$$

with

$$\left.\begin{aligned} &A_j = \alpha' a_j, \quad A_i = \alpha' a_i, \quad \alpha' = \frac{1}{\alpha} = \sec\left(\frac{\gamma}{2}\right), \\ &\beta' = \frac{1}{\beta} = \operatorname{cosec}\left(\frac{\gamma}{2}\right), \quad \epsilon = \frac{\alpha}{\beta}, \\ &C = \alpha'(x + x'), \quad D = \beta'(x - x'), \end{aligned}\right\} \quad (1.13.24)$$

a_j, a_i being the damping parameters. A detailed study is given in Heinzel (1981, 1982), Hubený (1982), Heinzel and Hubený (1983), Hubený *et al.* (1983).