Chapter 1

Definitions of fundamental quantities of the radiation field

1.1 Specific intensity

This is the most fundamental quantity of the radiation field. We shall be dealing with this quantity throughout this book.

Let dE_ν be the amount of radiant energy in the frequency interval $(\nu, \nu + d\nu)$ transported across an element of area ds and in the element of solid angle $d\omega$ during the time interval dt. This energy is given by

$$dE_\nu = I_\nu \cos\theta \, d\nu \, d\sigma \, d\omega \, dt, \tag{1.1.1}$$

where θ is the angle that the beam of radiation makes with the outward normal to the area ds, and I_ν is the *specific intensity* or simply *intensity* (see figure 1.1).

The dimensions of the intensity are, in CGS units, erg cm^{-2} s^{-1} hz^{-1} ster^{-1}. The intensity changes in space, direction, time and frequency in a medium that absorbs

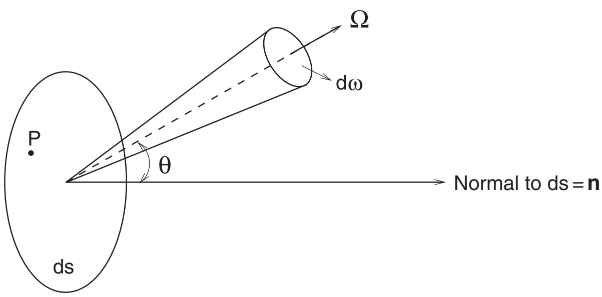

Figure 1.1 Schematic diagram which shows how the specific intensity is defined.

and emits radiation. I_ν can be written as

$$I_\nu = I_\nu(\mathbf{r}, \Omega, t), \tag{1.1.2}$$

where $\mathbf{r}$ is the position vector and Ω is the direction. In Cartesian coordinates it can be written as

$$I_\nu = I_\nu(x, y, z; \alpha, \beta, \gamma; t), \tag{1.1.3}$$

where x, y, z are the Cartesian coordinate axes and α, β, γ are the direction cosines. If the medium is stratified in plane parallel layers, then

$$I_\nu = I_\nu(z, \theta, \varphi; t), \tag{1.1.4}$$

where z is the height in the direction normal to the plane of stratification and θ and φ are the polar and azimuthal angles respectively. If I_ν is independent of φ, then we have a radiation field with axial symmetry about the z-axis. Instead of z, we may choose symmetry around the x-axis.

In spherical symmetry, I_ν is

$$I_\nu = I_\nu(r, \theta; t), \tag{1.1.5}$$

where r is the radius of the sphere and θ is the angle made by the direction of the ray with the radius vector.

The radiation field is said to be isotropic at a point, if the intensity is independent of direction at that point and then

$$I_\nu = I_\nu(\mathbf{r}, t). \tag{1.1.6}$$

If the intensity is independent of the spatial coordinates and direction, the radiation field is said to be homogeneous and isotropic. If the intensity I_ν is integrated over all the frequencies, it is called the integrated intensity I and is given by

$$I = \int_0^\infty I_\nu \, d\nu. \tag{1.1.7}$$

There are other parameters that characterize the state of polarization in a radiation field. These are studied in chapters 11 and 12.

1.2 Net flux

The flux F_ν is the amount of radiant energy transferred across a unit area in unit time in unit frequency interval. The amount of radiant energy in the area ds in the direction θ (see figure 1.1) to the normal, in the solid angle $d\omega$, in time dt and in the frequency interval $(\nu, \nu + d\nu)$ is equal to $I_\nu \cos\theta \, d\omega \, d\nu \, ds \, dt$. The net flow in all directions is

$$d\nu \, ds \, dt \int I_\nu \cos\theta \, d\omega,$$

or

$$F_\nu = \int I_\nu \cos\theta \, d\omega. \tag{1.2.1}$$

The integration is over all solid angles. This is the net flux and is the rate of flow of radiant energy per unit area per unit frequency.

In polar coordinates, where the outward normal is in the z-direction, we have

$$d\omega = \sin\theta \, d\theta \, d\varphi, \tag{1.2.2}$$

where φ is the azimuthal angle. The net flux F_ν then becomes

$$F_\nu = \int_0^{2\pi} \int_0^{\pi} I_\nu \cos\theta \sin\theta \, d\varphi \, d\theta. \tag{1.2.3}$$

The dimensions of flux are erg cm^{-2} s^{-1} hz^{-1}. Equation (1.2.3) can also be written as

$$\begin{aligned} F_\nu &= \int_0^{2\pi} d\varphi \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta + \int_0^{2\pi} d\varphi \int_{\pi/2}^{\pi} I_\nu \cos\theta \sin\theta \, d\theta \\ &= F_\nu(+) - F_\nu(-), \end{aligned} \tag{1.2.4}$$

where

$$F_\nu(+) = \int_0^{2\pi} \int_0^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta \, d\varphi \tag{1.2.5}$$

and

$$F_\nu(-) = \int_0^{2\pi} \int_{\pi}^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta \, d\varphi. \tag{1.2.6}$$

The physical meaning of equation (1.2.4) is as follows: $F_\nu(+)$ represents the radiation illuminating the area from one side and $F_\nu(-)$ represents the radiation illuminating the area from another side. Therefore F_ν, the flux of radiation transported through the area, is the difference between these illuminations of the area. The flux depends on the direction of the normal to the area. The dependence of the flux on direction shows that flux is of vector character. In the Cartesian coordinate system, let the angles made by the direction of radiation with the axes x, y and z be α_1, β_1 and γ_1 respectively, then the flux or radiation along the coordinate axes is given by

$$F_\nu(x) = \int I_\nu \cos\alpha_1 \, d\omega, \tag{1.2.7}$$

$$F_\nu(y) = \int I_\nu \cos\beta_1 \, d\omega, \tag{1.2.8}$$

$$F_\nu(z) = \int I_\nu \cos\gamma_1 \, d\omega. \tag{1.2.9}$$

Furthermore, if α_2, β_2 and γ_2 are the angles made by the coordinate axes and the normal to the area and θ is the angle between the normal and the direction of the radiation, then

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \tag{1.2.10}$$

Substituting equation (1.2.10) into equation (1.2.1), we get

$$F_\nu = \cos\alpha_2 \, F_\nu(x) + \cos\beta_2 \, F_\nu(y) + \cos\gamma_2 \, F_\nu(z). \tag{1.2.11}$$

The integrated flux over frequency is

$$F = \int_0^\infty F_\nu \, d\nu. \tag{1.2.12}$$

If the radiation field is symmetric with respect to the coordinate axes, then the net flux across the surface oriented perpendicular to that axis is zero as the oppositely directed rays cancel each other. In a homogeneous planar geometry, $F_\nu(x)$ and $F_\nu(y)$ are zeros and only $F_\nu(z)$ exists. In such a situation, we have

$$F_\nu(z, t) = 2\pi \int_{-1}^{+1} I(z, \mu, t)\mu \, d\mu, \tag{1.2.13}$$

where $\mu = \cos\theta$.

The astrophysical flux $F_{A\nu}(z, t)$ normally absorbs the π on the RHS of equation (1.2.13) and is written as

$$F_{A\nu}(z, t) = 2 \int_{-1}^{+1} I(z, \mu, t)\mu \, d\mu \tag{1.2.14}$$

and the Eddington flux $F_{E\nu}$ is defined as

$$F_{E\nu}(z, t) = \frac{1}{2} \int_{-1}^{+1} I(z, \mu, t)\mu \, d\mu. \tag{1.2.15}$$

1.2.1 Specific luminosity

The specific luminosity was suggested by Rybicki (1969) and Kandel (1973). We define it following Collins (1973).

From figure 1.2, we define the specific luminosity $\mathcal{L}(\psi, \xi)$ in terms of the orientation variables ψ and ξ as

$$\mathcal{L}(\psi, \xi) = 4\pi \int_A I(\theta, \phi)\hat{n}(\theta, \phi) \cdot \hat{o}(\theta, \phi) \, dA(\theta, \phi), \tag{1.2.16}$$

where $\hat{n}(\theta, \phi)$ and $\hat{o}(\theta, \phi)$ are position dependent unit vectors normal to the surface and in the direction of the observer respectively. The area A over which the specific intensity $I(\theta, \phi)$ is to be integrated is the 'observable' surface and is defined by the orientation angles ψ and ξ. It is obvious from equation (1.2.16) that $\mathcal{L}(\psi, \xi)$ is a function of the orientation of the object with respect to the observer and is measured per unit solid angle; the total luminosity L is given in terms of $\mathcal{L}(\psi, \xi)$ as

$$L = \frac{1}{4\pi} \int_{4\pi} \mathcal{L}(\psi, \xi)\, d\Omega(\psi, \xi). \tag{1.2.17}$$

1.3 Density of radiation and mean intensity

Let V and Σ be two regions (see figure 1.3) the latter being larger than the former in linear dimensions but sufficiently small for a pencil not to have its intensity changed appreciably in transit. The radiation travelling through V must have crossed the region Σ through some element; let $d\Sigma$ be such an element with normal **N**. The

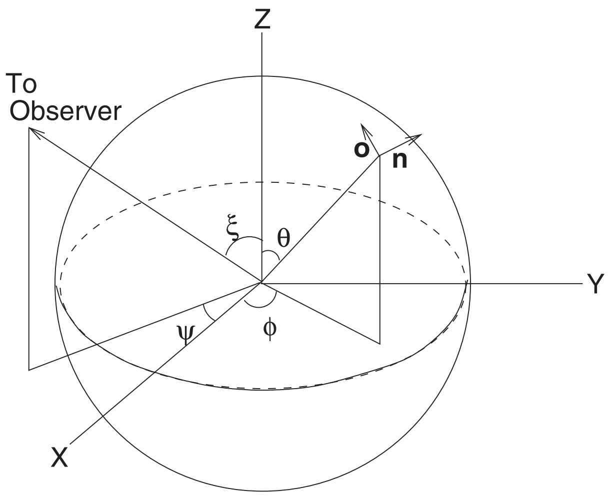

Figure 1.2 The angles θ and ϕ are the angular coordinates of a point on the stellar surface, and therefore represent a local structure. The angles ψ and ξ represent the orientation of the stellar body (from Collins (1973), with permission).

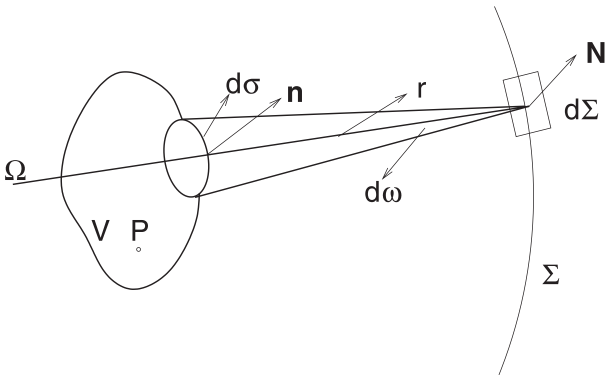

Figure 1.3 Schematic diagram to define density of radiation.

energy passing through $d\Sigma$ which also passes through $d\sigma$ with normal $\mathbf{n}$ on V per unit time is

$$I_\nu(\mathbf{\Omega}, \mathbf{N})\, d\Sigma\, d\omega'\, d\nu, \tag{1.3.1}$$

where

$$d\omega' = (\mathbf{\Omega} \cdot \mathbf{n})\, d\sigma / r^2. \tag{1.3.2}$$

If l is the length travelled by the pencil in V, then an amount of energy

$$\frac{I_\nu(\mathbf{\Omega} \cdot \mathbf{n})(\mathbf{\Omega} \cdot \mathbf{N})\, d\sigma\, d\Sigma\, d\nu}{r^2} \frac{l}{c} \tag{1.3.3}$$

will have travelled through the element in time l/c, where c is the velocity of light.

The solid angle $d\omega$ subtended by $d\Sigma$ at P is $(\Omega \cdot \mathbf{N})\, d\Sigma / r^2$ and the volume intercepted in V by the pencil is given by

$$dV = l(\mathbf{\Omega} \cdot \mathbf{n})\, d\sigma. \tag{1.3.4}$$

This amount of energy is given by

$$\frac{1}{c} I_\nu\, d\nu\, dV\, d\omega. \tag{1.3.5}$$

Therefore, the contribution to the energy per unit volume per unit frequency range (in the interval ν, $\nu + d\nu$) coming from the solid angle $d\omega$ about the direction Ω is $I_\nu\, d\omega / c$ and the energy density is defined as

$$U_\nu = \frac{1}{c} \int I_\nu\, d\omega. \tag{1.3.6}$$

The average intensity or mean intensity J_ν is

$$J_\nu = \frac{1}{4\pi} \int I_\nu\, d\omega, \tag{1.3.7}$$

so that

$$U_\nu = \frac{4\pi}{c} J_\nu. \tag{1.3.8}$$

For an axially symmetric radiation field, J_ν is given by

$$\begin{aligned} J_\nu &= \frac{1}{2} \int_0^\pi I_\nu \sin\theta\, d\theta \\ &= \frac{1}{2} \int_{-1}^{+1} I(\mu)\, d\mu. \end{aligned} \tag{1.3.9}$$

The integrated energy density U is

$$U = \int_0^\infty U_\nu\, d\nu = \frac{1}{c} \int I\, d\omega. \tag{1.3.10}$$

The dimensions of energy density are erg cm^{-3} hz^{-1} and those of the integrated energy density are erg cm^{-3}. The dimensions of the mean intensity are erg cm^{-2} s^{-1} hz^{-1}.

1.4 Radiation pressure

A quantum of energy $h\nu$ will have a momentum of $h\nu/c$, where c is the velocity of light in the direction of propagation. The pressure of radiation at the point P (see figure 1.1) is calculated from the net rate of transfer of momentum normal to an area ds, which contains the point P. The amount of radiant energy in the frequency range $(\nu, \nu + d\nu)$ incident on ds making an angle θ with the normal to ds traversing the solid angle $d\omega$ in time dt is

$$I_\nu \cos\theta \, d\omega \, d\nu \, ds \, dt. \tag{1.4.1}$$

The momentum associated with this energy in the direction I_ν is

$$\frac{1}{c} I_\nu \cos\theta \, d\omega \, d\nu \, ds \, dt. \tag{1.4.2}$$

Therefore the normal component of the momentum transferred across ds by the radiation is

$$\frac{1}{c} \, d\sigma \, dt \, I_\nu \cos^2\theta \, d\omega \, dt. \tag{1.4.3}$$

The net transfer of momentum across ds by the radiation in the frequency interval $(\nu, \nu + d\nu)$ is

$$\frac{d\sigma \, dt}{c} \int I_\nu \cos^2\theta \, d\omega \, d\nu, \tag{1.4.4}$$

where the integration is over the whole sphere. The pressure at the point P is the net rate of transfer of momentum normal to the element of the surface area containing P in the unit area; the pressure $p_r(\nu) \, d\nu$ can be written in the frequency interval as

$$p_r(\nu) = \frac{1}{c} \int_0^{2\pi} \int_0^{\pi} I_\nu \cos^2\theta \sin\theta \, d\theta \, d\varphi. \tag{1.4.5}$$

If the radiation field is isotropic, then

$$p_r(\nu) = \frac{2\pi}{c} I_\nu \int_0^{\pi} \mu^2 \, d\mu = \frac{4}{3} \frac{\pi}{c} I_\nu \qquad (\mu = \cos\theta) \tag{1.4.6}$$

or in terms of energy density U_ν

$$p_r(\nu) = \frac{1}{3} U_\nu. \tag{1.4.7}$$

The radiation pressure integrated over all frequencies is

$$p_r = \int_0^{\infty} p_r(\nu) \, d\nu \tag{1.4.8}$$

or

$$p_r = \frac{1}{c} \int I \cos^2\theta \, d\omega, \tag{1.4.9}$$

where I is the integrated intensity. Furthermore

$$p_r = \frac{1}{3}U. \tag{1.4.10}$$

It can be seen that the dimensions of radiation pressure are the same as those of energy density, that is, erg cm^{-3} hz^{-1} and the integrated radiation pressure has the dimensions of erg cm^{-3}.

1.5 Moments of the radiation field

Moments are defined in such a way that the nth moment over the radiation field is given by

$$M_n(z, n) = \frac{1}{2}\int_{-1}^{+1} I_\nu(z, \mu)\mu^n \, d\mu. \tag{1.5.1}$$

Following Eddington, we can have the zeroth, first and second moments as:

1. Zeroth moment (mean intensity):

$$J_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I(z, \mu) \, d\mu. \tag{1.5.2}$$

2. First moment (Eddington flux):

$$H_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I(z, \mu)\mu \, d\mu. \tag{1.5.3}$$

3. Second moment (the so called K-integral):

$$K_\nu(z) = \frac{1}{2}\int_{-1}^{+1} I(z, \mu)\mu^2 \, d\mu. \tag{1.5.4}$$

1.6 Pressure tensor

The rate of transfer of the x-component of the momentum across the element of surface normal to the x-direction by radiation in the solid angle dw per unit area in the direction whose direction cosines are l, m, n is

$$\frac{1}{c} I l \, d\omega \, l, \tag{1.6.1}$$

where I is the integrated radiation. If monochromatic radiation is considered, then I should be replaced by $I_\nu \, d\nu$. The total rate of x-momentum transfer across the element per unit area is $p_r(xx)$:

$$p_r(xx) = \frac{1}{c}\int I l^2\, d\omega. \tag{1.6.2}$$

Similarly the y- and z-components are given by

$$p_r(xy) = \frac{1}{c}\int I l m\, d\omega \quad \text{and} \quad p_r(xz) = \frac{1}{c}\int I l n\, d\omega. \tag{1.6.3}$$

The quantities $p_r(yx)$, $p_r(yy)$, $p_r(yz)$, $p_r(zx)$, $p_r(zy)$ and $p_r(zz)$ are similarly defined for elements of the surfaces normal to the y- and z-directions. These nine quantities constitute the 'stress tensor'.

One can see that $p_r(xy) = p_r(yx)$, $p_r(xz) = p_r(zx)$ and $p_r(yz) = p_r(zy)$ or that the tensor is symmetrical. The mean pressure $\bar{p}$ is defined by

$$\bar{p} = \frac{1}{3}\left[p_r(xx) + p_r(yy) + p_r(zz)\right], \tag{1.6.4}$$

and

$$\bar{p} = \frac{1}{3c}\int I\omega = \frac{1}{3}U, \tag{1.6.5}$$

as $l^2 + m^2 + n^2 = 1$.

In the case of an isotropic radiation field

$$\bar{p} = p_r(xx) = p_r(yy) = p_r(zz) = \frac{1}{3}U, \tag{1.6.6}$$

and

$$\left.\begin{aligned} p_r(xy) &= p_r(yx) = 0,\\ p_r(xz) &= p_r(zx) = 0,\\ p_r(yz) &= p_r(xy) = 0. \end{aligned}\right\} \tag{1.6.7}$$

1.7 Extinction coefficient: true absorption and scattering

A pencil of radiation of intensity I_ν is attenuated while passing through matter of thickness ds and its intensity becomes $I_\nu + dI_\nu$, where

$$dI_\nu = -I_\nu \kappa_\nu\, ds. \tag{1.7.1}$$

The quantity κ_ν is called the mass extinction coefficient or the mass absorption coefficient. κ_ν comprises two important processes: (1) true absorption and (2) scattering. Therefore we can write

$$\kappa_\nu = \kappa_\nu^a + \sigma_\nu, \tag{1.7.2}$$

where κ_ν^a and σ_ν are the absorption and scattering coefficients respectively. Absorption is the removal of radiation from the pencil of the beam by a process which involves changing the internal degrees of freedom of an atom or a molecule. Examples of these processes are: (1) photoionization or bound–free absorption by which the photon is absorbed and the excess energy, if any, goes into the kinetic energy of the electron thermalizing the medium; (2) the absorption of a photon by a freely moving electron that changes its kinetic energy which is known as free–free absorption; (3) the absorption of a photon by an atom leading to excitation from one bound state to another bound state, which is called bound–bound absorption or photoexcitation; (4) the collision of an atom in a photoexcited state which will contribute to the thermal pool; (5) the photoexcitation of an atom which ultimately leads to fluorescence; (6) negative hydrogen absorption, etc. The reversal of the above processes may contribute to the emission coefficient (see section 1.8).

The coefficient κ_ν^a depends on the thermodynamic state of the matter at (pressure p, temperature T, chemical abundances α_i) any given point in the medium. At the point r the coefficient is given by

$$\kappa_\nu^a(r, T) = \kappa_\nu^a\,[p(r, T), T(r), \alpha_i(r, T), \ldots, \alpha_\kappa(r, T)], \tag{1.7.3}$$

when there is local thermodynamic equilibrium (LTE). This kind of situation does not exist in reality and one needs to determine the κ_ν^a in a non-LTE situation. In static media κ_ν^a is isotropic while in moving media it is angle and frequency dependent due to Doppler shifts.

Another process by which energy is lost from the beam is the scattering of radiation which is represented by the mass scattering coefficient κ_ν^s. Scattering changes not only the photon's direction but also its energy. If we define the *albedo for single scattering* as ω_ν, then

$$\omega_\nu = \frac{\sigma_\nu}{\kappa_\nu}, \tag{1.7.4}$$

is the ratio of scattering to the extinction coefficients.

The extinction coefficient is the product of the atomic absorption coefficients or scattering coefficients (cm^2) and the number density of the absorbing or scattering particles (cm^{-3}). The dimension of κ_ν is cm^{-1} and $1/\kappa_\nu$ gives the photon mean free path which is the distance over which a photon travels before it is removed from the pencil of the beam of radiation.

1.8 Emission coefficient

Let an element of mass with a volume element dV emit an amount of energy dE_ν into an element of solid angle $d\omega$ centred around $\mathbf{\Omega}$ in the frequency interval ν to $\nu + d\nu$ and time interval t to $t + dt$. Then

$$dE_\nu = j_\nu\, dV\, d\omega\, d\nu\, dt, \tag{1.8.1}$$