1

Small-Scale Statistics and Structure of Turbulence – in the Light of High Resolution Direct Numerical Simulation

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1.1 Introduction

Fully developed turbulence is a phenomenon involving huge numbers of degrees of dynamical freedom. The motions of a turbulent fluid are sensitive to small differences in flow conditions, so though the latter are seemingly identical they may give rise to large differences in the motions.[1] It is difficult to predict them in full detail.

This difficulty is similar, in a sense, to the one we face in treating systems consisting of an Avogadro number of molecules, in which it is impossible to predict the motions of them all. It is known, however, that certain relations, such as the ideal gas laws, between a few number of variables such as pressure, volume, and temperature are insensitive to differences in the motions, shapes, collision processes, etc. of the molecules.

Given this, it is natural to ask whether there is any such relation in turbulence. In this regard, we recall that fluid motion is determined by flow conditions, such as boundary conditions and forcing. It is unlikely that the motion would be insensitive to the difference in these conditions, especially at large scales. It is also tempting, however, to assume that, in the statistics at sufficiently small scales in fully developed turbulence at sufficiently high Reynolds number, and away from the flow boundaries, there exist certain kinds of relation which are universal in the sense that they are insensitive to the detail of large-scale flow conditions. In fact, this idea underlies Kolmogorov's theory (Kolmogorov, 1941a, hereafter referred as K41), and has been at the heart of many modern studies of turbulence. Hereafter, universality in this sense is referred to as universality in the sense of K41.

Although most of the energy in turbulence resides at large scales, most of

[a] This work was undertaken while both authors were at Nagoya University.
[1] This does not prevent satisfactory averages being measured, at least those belonging to small scales.

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the degrees of dynamical freedom resides in the small scales. In Fourier space, for example, most of the Fourier modes are in the high-wavenumber range. Hence properly understanding the nature of turbulence at small scales is interesting, not only from the theoretical, but also from the practical point of view, because such an understanding can be expected to be useful for developing models of turbulence to properly reduce the degrees of freedom to be treated.

This chapter will review studies of the nature of turbulence at small scales. Of course, more intensive studies have been performed on this interesting subject than we can cover here; in addition, we cannot review all of the issues related to each study that we do cover. We present a review of a few topics in the light of recent progress in high resolution direct numerical simulation (DNS) of turbulence. An analysis is also made on elongated local eddy structure and statistics. An emphasis is placed upon the Reynolds number dependence of the statistics and on the difference between active and non-active regions in turbulence.

1.2 Background supporting the idea of universality

1.2.1 Kolmogorov's 4/5 law

The existence of universality in the sense of K41 has not yet been proven rigorously, but there is evidence supporting it. Among this is Kolmogorov's 4/5 law (Kolmogorov, 1941c), which is derived as a consequence of the Navier–Stokes (NS) equation governing fluid motion.

Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be an incompressible turbulent velocity field obeying the Navier–Stokes equation,

$$\frac{\partial}{\partial t}\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}, \tag{1.1}$$

and the incompressibility condition,

$$\nabla \cdot \mathbf{u} = \mathbf{0}, \tag{1.2}$$

where ν is the kinematic viscosity, p the pressure, $\mathbf{f}$ the external force per unit mass, and ρ the fluid density.

For homogeneous and isotropic (HI) turbulence, the NS equation with the incompressibility condition (1.2) yields the Kármán–Howarth equation (Kármán and Howarth, 1938)

$$B_3^L(r) = -\frac{4}{5}\langle\epsilon\rangle r + 6\nu\frac{\partial B_2^L(r)}{\partial r} + F(r) - \frac{3}{r^4}\int_0^r \frac{\partial B_2^L(\tilde{r})}{\partial t}\tilde{r}^4 d\tilde{r}, \tag{1.3}$$

where $\langle\epsilon\rangle$ is the average of the rate of energy dissipation ϵ per unit mass, and $B_n^L(r)$ is the nth order structure function of the longitudinal velocity difference δu^L defined as

$$B_n^L(r) \equiv \left\langle \left[\delta u^L(r)\right]^n \right\rangle, \quad \delta u^L(r) \equiv [\mathbf{u}(\mathbf{x}+r\mathbf{e}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{e}, \tag{1.4}$$

in which $\mathbf{e}$ is an arbitrary unit vector. In (1.3), F is expressed in terms of the correlation $g(r) \equiv \langle[\mathbf{u}(\mathbf{r}+\mathbf{x}) - \mathbf{u}(\mathbf{x})] \cdot [\mathbf{f}(\mathbf{r}+\mathbf{x}) - \mathbf{f}(\mathbf{x})]\rangle$. It is shown by simple algebra that

$$F(r) = \frac{6}{r^4}\int_0^r \tilde{r}G(\tilde{r})d\tilde{r}, \quad G(r) = \int_0^r \tilde{r}^2 g(\tilde{r})d\tilde{r}.$$

If the forcing $\mathbf{f}$ is confined only to large scales, say $\sim L$ (where the symbol $\sim$ denotes an equality up to a coefficient of order unity), and the viscosity ν is very small, then it is plausible to assume that in (1.3),

(i) the forcing term $F(r)$ is negligible at $r \ll L$,
(ii) the viscosity term works only at small scales, say $\sim \eta$, so that it is negligible at $r \gg \eta$, and
(iii) the statistics is almost stationary at small scales, so that the last term is negligible at $r \ll L$.

Under these assumptions, (1.3) yields the 4/5 law,

$$B_3^L(r) = -\frac{4}{5}\langle\epsilon\rangle r, \tag{1.5}$$

for $L \gg r \gg \eta$.

Note that the 4/5 law (1.5) applies not only to the stationary but also to the freely-decaying case, as long as one may assume (iii), in addition to (i) and (ii), where L is to be understood appropriately, e.g., as the characteristic length scale of the energy containing eddies.

The relation (1.5) asserts that $B_3^L(r)$ is specified only by $\langle\epsilon\rangle$ and r. It holds independently of the shapes, internal structures, deformations, positions, alignments, interactions, collision and reconnection processes, etc. of small-scale eddies, however the term 'eddies' may be defined, and also of the forcing and boundary conditions outside the range $r \ll L$, as long as (i), (ii) and (iii) hold. (This doesn't mean that $B_3^L(r)$ is independent of these factors and conditions, as they may still affect $\langle\epsilon\rangle$: rather, the relation (1.5) means that their influence, if any, is only through $\langle\epsilon\rangle$.)

The relation (1.5) holds independently of these factors, just as the ideal gas laws hold independently of the shapes, internal structures, interactions, collision processes etc. of the molecules comprising the gas, and independently of the shape of the container of the gas. The relation is in this sense

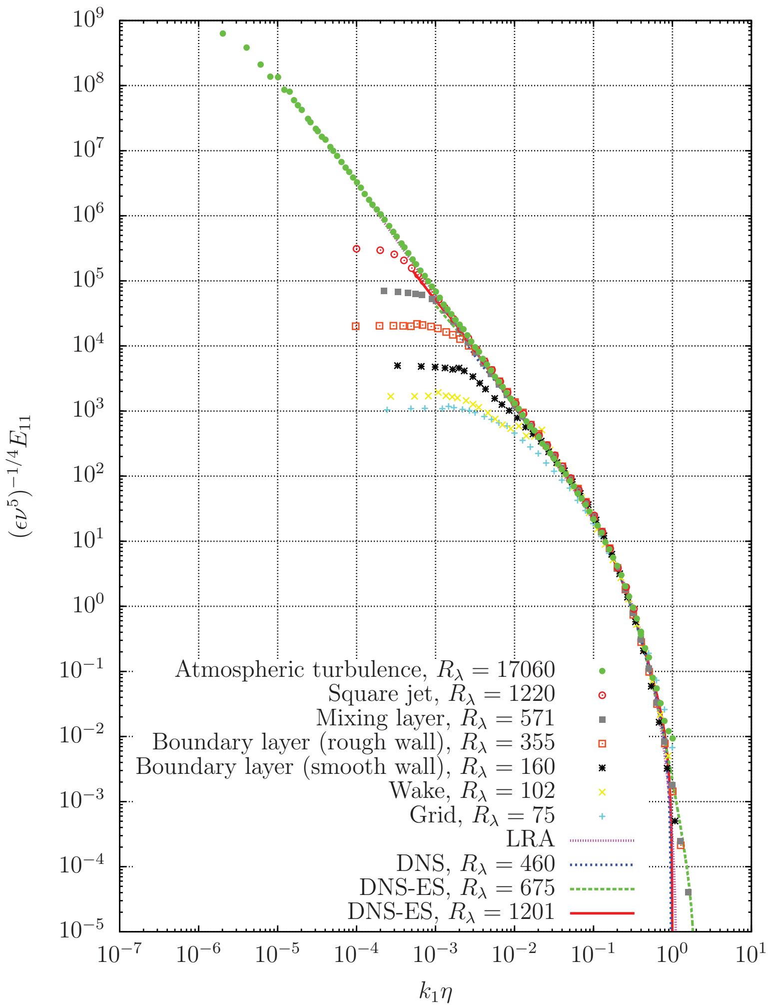

Figure 1.1 Normalized longitudinal one-dimensional energy spectrum. The data except those by DNS-ES are re-plotted from Tsuji (2009).

universal, and supports the idea of existence of universality in the sense of K41.

1.2.2 Energy spectrum

More support for the existence of universality in the sense of K41 is given by the second-order two-point velocity correlations, or, equivalently, the velocity correlation spectra observed in experiments and DNS. If the second-order moments of $\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ are universal in a certain sense at small scale,

$r \ll L$, then so are their spectra, i.e., their Fourier transforms with respect to $\mathbf{r}$ at large $k \gg 1/L$. The converse is also true. Here L is the characteristic length scale of the energy containing eddies. The universality of these statistics is at the heart of Kolmogorov's theory, K41 (see below). Experimental and DNS data have been accumulated, including those of the well-known tidal channel observation by Grant *et al.* (1962), according to which the spectra under different flow conditions overlap well at high wavenumbers under appropriate normalization (see, e.g., Monin and Yaglom, 1975).

Figure 1.1 illustrates collections of energy spectra. It shows the longitudinal one-dimensional energy spectrum E_{11} vs. $k_1\eta$ in turbulent mixing layers, boundary layers and atmospheric turbulence, at Taylor micro-scale Reynolds numbers R_λ up to $\approx 17,000$. Here E_{11} satisfies

$$\langle u_1^2 \rangle = \int_0^\infty E_{11}(k_1)dk_1,$$

in which $\langle u_1^2 \rangle$ and k_1 are respectively the mean-square fluctuation velocity and the wavenumber component in the longitudinal direction, $\eta \equiv (\nu^3/\langle \epsilon \rangle)^{1/4}$ is the Kolmogorov length scale, and $R_\lambda \equiv u'\lambda/\nu$ with $3u'^2/2 = E$ being the total kinetic energy of the fluctuating velocity per unit mass, and $\lambda \equiv (15\nu u'^2/\langle \epsilon \rangle)^{1/2}$ is the Taylor micro-scale. It also shows the spectra by DNS ($R_\lambda = 460$) by Gotoh *et al.* (2002) and a series of DNS performed on the Earth Simulator, hereafter referred to as DNS-ES, with R_λ and the number of grid points up to approximately 1,200 and 4096^3, respectively (Yokokawa *et al.*, 2002; Kaneda *et al.*, 2003). DNS with grid points as large as 4096^3 has been also used in studies of turbulence fields, see for example Donzis and Sreenivasan (2010) and Donzis *et al.* (2010).

In spite of the different flow conditions, the spectra are seen to overlap well at large $k_1\eta$, at least to the extent visible in the figure. This supports the idea that there may be certain kinds of relations which are insensitive to the details of flow conditions at large scales.

The overlap is in agreement with Kolmogorov's K41 theory, according to which $E_{11}(k_1)$ is of the form

$$E_{11}(k_1)/(\langle \epsilon \rangle \nu^5)^{1/4} = \phi_{11}(k_1\eta), \tag{1.6}$$

in the wavenumber range $k \gg k_L \equiv 1/L$, and in particular

$$E_{11}(k_1) \approx C_{\mathrm{K}} \langle \epsilon \rangle^{2/3} k_1^{-5/3}, \tag{1.7}$$

in the inertial subrange $k_L \ll k_1 \ll k_d$, where $k_d \equiv 1/\eta$, the universal function ϕ_{11} depends only on $k_1\eta$, and where C_{K} is a dimensionless universal constant. In terms of the three-dimensional energy spectrum $E(k)$, (1.7) is equivalent to

$$E(k) \approx K_{\rm o} \langle \epsilon \rangle^{2/3} k^{-5/3}, \quad \text{with } K_{\rm o} = \frac{55}{18} C_{\rm K}, \tag{1.8}$$

where k is the three-dimensional wavenumber. In physical space, (1.7) is equivalent to

$$B_2^L(r) \approx C_{\rm R} \langle \epsilon \rangle^{2/3} r^{2/3}, \quad \text{with } C_{\rm R} = \frac{3}{2} \Gamma \left(\frac{1}{3} \right) C_{\rm K}, \tag{1.9}$$

in the range $L \gg r \gg \eta$. The scaling $r^{2/3}$ in (1.9), as well as $k^{-5/3}$ in (1.7) and (1.8), can be derived from the 4/5 law and by assuming the skewness

$$\left\langle [\delta u^L(r)]^3 \right\rangle / \left\langle [\delta u^L(r)]^2 \right\rangle^{3/2}$$

to be constant in the inertial subrange (Kolmogorov, 1941c).

The experimental and DNS data reported thus far suggest that $C_{\rm K} \approx 0.5 \pm 0.05$ (see, e.g., Sreenivasan, 1995; Sreenivasan and Antonia, 1997), although a close inspection of the spectra shows that they don't agree with (1.7) in a strict sense. Readers may refer to e.g., Tsuji (2009) for a review of studies on this discrepancy.

One can consider at least two kinds of origins for the discrepancy:

(a) the inertial range intermittency;
(b) the behavior of the spectrum around $k\eta \sim 1$, where the compensated spectrum $E(k)/(\langle \epsilon \rangle^{2/3} k^{-5/3})$ shows a "bump", known as bottleneck (see, e.g., the review in Donzis and Sreenivasan, 2010).

Such a bump is known to be more noticeable in the normalized spectrum of $E(k)$ than $E_{11}(k_1)$ (see, e.g., Kaneda and Ishihara, 2007). The bottleneck does not contradict (1.6), (1.7), or (1.8) under the conditions of K41. A simple closure, Obukhov's constant skewness model (Obukhov, 1949), captures the bottleneck (see p. 404 in Monin and Yaglom, 1975, or Fig. 6.19 in Davidson, 2004).

By using data from DNS with the number of grid points up to 4096^3 and $R_\lambda \approx 1{,}000$, Donzis and Sreenivasan (2010) addressed the difficulty in estimating the constants $K_{\rm o}$, and proposed a procedure for determining the constant by taking into account the effects (a) and (b), which yields $K_{\rm o} \approx 1.58$.

The experimental value $C_K \approx 0.5 \pm 0.05$, (i.e. $K_{\rm o} \approx 1.53 \pm 0.15$) is not far from the one derived by Lagrangian closure theories, the Abridged Lagrangian History Direct Interaction Approximation, ALDHIA, (Kraichnan, 1965, 1966), Strain based-LHDIA (Kraichnan and Herring, 1978; Herring and Kraichnan, 1979) and the Lagrangian Renormalized Approximation, LRA, (Kaneda, 1981; Gotoh *et al.*, 1988), which are fully consistent with K41, and can be obtained by the lowest-order truncations of systematic renormalized perturbative expansions without introducing any ad hoc adjusting parameter. They give $K_o \approx 1.77, 2.0$ and 1.72, respectively. The spectrum for $R_\lambda \to \infty$ by the LRA is also plotted in Fig. 1.1.

1.3 Examination of the ideas underlying the 4/5 law

1.3.1 Energy dissipation rate at $\nu \to 0$

Although plausible in itself, Kolmogorov's 4/5 law is derived on the basis of several assumptions. Among them is one concerned with the average $\langle\epsilon\rangle$ in the limit $\nu \to 0$, or equivalently in the limit of the Reynolds number $Re \equiv u'L/\nu \to \infty$. It is assumed that $\langle\epsilon\rangle$ is not so small that the first term is dominant on the right hand side of (1.3) for $L \gg r \gg \eta$, in the limit $\nu \to 0$. This assumption is concerned with a fundamental question on the smoothness of the turbulent field:

(a) does $\langle\epsilon\rangle$ remain non-zero finite;
(b) or does $\langle\epsilon\rangle \to 0$ in the limit?

If (a) is true, we may safely assume that the first term is dominant on the right-hand side of (1.3) under the conditions (i), (ii) and (iii) used in deriving (1.5). Since $\langle\epsilon\rangle = 2\nu\langle e_{ij}e_{ij}\rangle$, (a) implies that $\langle e_{ij}e_{ij}\rangle \to \infty$, i.e., the mean square of at least one component of the velocity gradient tensor diverges as $\nu \to 0$, where $e_{ij} \equiv (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, and we use the summation convention for repeated indices. The condition (a) also implies that the dissipation in the limit $\nu \to 0$ is different from that in the ideal fluid with $\nu = 0$, in which $\langle\epsilon\rangle$ must be zero, and in this sense the limit is singular. Such a singularity is well known for flow past a solid body: neglecting it gives rise to D'Alembert's paradox.

Let D_ϵ be the normalized dissipation rate defined by $D_\epsilon \equiv L\langle\epsilon\rangle/u'^3$. Rigorous upper bounds for D_ϵ have been derived; for example, it has been shown that, in body-forced turbulence,

$$D_\epsilon \le \frac{a}{Re} + b, \tag{1.10}$$

where the prefactors a and b depend only on the functional shape of the body force and not on its magnitude or any other length scales in the force, the domain or the flow, and $\langle\epsilon\rangle$ and L are to be understood as the time-average of ϵ and the longest length scale in the applied forcing function, respectively (Doering and Foias, 2002; Doering, 2009). As regards the lower band, an application of Poincaré's inequality gives $D_\epsilon \geq 4\pi^2/(\alpha^2 Re)$, as noted by Doering and Foias (2002), where $\alpha = L'/L$ is the ratio of the domain size L' to L. However, these inequalities are not sufficient to answer the question of whether (a) is true or not.

Studies have been made to address this question through DNS. The DNS data with R_λ up to approximately 200 compiled by Sreenivasan (1998), which include both decaying and forced turbulence data, shows that D_ϵ decreases with R_λ for $R_\lambda < 200$ or so, while adding the DNS-ES data up to $R_\lambda \approx 1200$ strongly suggests that D_ϵ remains finite and independent of Re in the limit of $Re \to \infty$ (Kaneda *et al.*, 2003).

The right-hand side of (1.10) decreases rapidly with Re at small Re, then decreases slowly at larger Re, and approaches a constant as $Re \to \infty$. The latter is consistent with the expectation that $D_\epsilon \to$ constant as $Re \to \infty$, and the former, or the first term in (1.10), is the result valid for asymptotically small Re, (Sreenivasan, 1984; Doering and Foias, 2002). By using the rigorous relation $Re = D_\epsilon R_\lambda^2/15$ (see (1.12) below), (1.10) can be cast into the form

$$D_\epsilon \leq \frac{b}{2}\left(1 + \sqrt{1 + \frac{4a}{b^2}\frac{1}{R_\lambda'^2}}\right), \tag{1.11}$$

where $R_\lambda' \equiv R_\lambda/\sqrt{15}$. Donzis *et al.* (2005) showed that the relation (1.11), with the inequality replaced by equality, provides a good fit for the Reynolds number dependence of D_ϵ in DNS. Another expression of the Reynolds number dependence of D_ϵ was proposed by Lohse (1994) on the basis of physical assumptions. Readers may refer to Tsinober (2009) and references cited therein for more DNS and experimental studies on D_ϵ.

Simple algebra gives

$$\frac{L}{\eta} = D_\epsilon^{1/2} Re^{3/4} = 15^{-3/4} D_\epsilon^{5/4} R_\lambda^{3/2} \propto Re^{3/4}, \tag{1.12}$$

where the last proportionality is satisfied if D_ϵ is independent of Re.

1.3.2 *Influence of finite* Re, L/r *and* r/η

The 4/5 law assumes that Re, L/r and r/η are sufficiently large. The same is true in K41, which assumes $Re, L/r \to \infty$. Even if these theories are correct, they themselves do not provide any quantitative answer to the question of how large Re, L/r and r/η must be for the theories to achieve any required accuracy. In this respect (1.3) is interesting, because it is an exact relation (subject to the homogeneity and isotropy conditions), and may give a quantitative answer to such a question.

As regards the forcing term in (1.3), it can be shown by simple algebra that $F(r)/(\langle\epsilon\rangle r) \approx C_{\rm f}(r/L)^2$, provided that the forcing is exerted only at low wavenumber modes, as in DNS-ES, where $C_{\rm f}$ is a dimensionless constant of order unity which may depend on the method of forcing (see, for example (Gotoh *et al.*, 2002) and (Kaneda *et al.*, 2008)).

The viscosity term is not easy to evaluate rigorously, but a simple model based on (1.8) may be applicable as a first approximation (Lindborg, 1999; Moisy *et al.*, 1999; Qian, 1999; Lundgren, 2003; Davidson, 2004; Kaneda *et al.*, 2008, the last of which is referred to below as KYI). Substituting (1.9) into the viscosity term of (1.3) and assuming statistical stationarity gives

$$\Delta(r) \equiv \frac{B_3^L(r)}{\langle\epsilon\rangle\, r} + \frac{4}{5} = C_{\rm v}\left(\frac{r}{\eta}\right)^{-4/3} + C_{\rm f}\left(\frac{r}{L}\right)^2, \tag{1.13}$$

where $C_{\rm v} = (44/9)C_{\rm R}$. Although this model is not rigorous, especially outside the inertial subrange, it was confirmed to be in good agreement with DNS-ES at large R_λ (>400 or so) and $L \gg r \gg \eta$ (KYI; Ishihara *et al.*, 2009).

Equation (1.13) implies that the difference $\Delta(r)$ takes its minimum $\Delta_{\rm min}$ as function of r at $r = r_m$, where

$$\Delta_{\rm min} \propto R_\lambda^{-6/5}, \qquad \frac{r_m}{\eta} \propto \left(\frac{L}{\eta}\right)^{3/5} \propto R_\lambda^{9/10}, \tag{1.14}$$

and the last proportionality is satisfied when (1.12) holds (see, for example, (Moisy *et al.*, 1999; Qian, 1999) and KYI).

For decaying turbulence in the absence of external forcing $\mathbf{f}$, one may insert $F = 0$ in (1.3). But in this case its last term is not zero because turbulence cannot be stationary without external forcing. Lindborg (1999) showed that a simple model based on (1.9) and K–ϵ type modeling *à la* Kolmogorov (1941b) gives

$$\Delta(r) = C_{\rm v}\left(\frac{r}{\eta}\right)^{-4/3} + C_n R_\lambda^{-1}\left(\frac{r}{\eta}\right)^{2/3}, \tag{1.15}$$

(C_n is a constant), so that $\Delta(r)$ takes its minimum $\Delta_{\rm min}$ at $r = r_{\rm m}$, where

$$\Delta_{\rm min} \propto R_\lambda^{-2/3}, \qquad \frac{r_{\rm m}}{\eta} \propto R_\lambda^{1/2}. \tag{1.16}$$

The approximation (1.15) can be derived also by a method of matched asymptotic expansion (Lundgren, 2003).

The decay of $\Delta_{\min}$ in proportion to $R_\lambda^{-6/5}$ in (1.14) for forced turbulence, and in proportion to $R_\lambda^{-2/3}$ in (1.16) for decaying turbulence, is in agreement with the data compiled by Antonia and Burattini (2006). The estimates (1.15) and (1.16) are consistent with those by Qian (1999). Decay in proportion to $R_\lambda^{-6/5}$ in forced turbulence was also confirmed to be in agreement with the DNS-ES data (KYI). For experimental or DNS data of $B_3^L(r)$, or for the comparison of (1.13) or (1.15) with experimental or DNS data, readers may refer to, for example, (Watanabe and Gotoh, 2004; Antonia and Burattini, 2006) and references cited therein, in addition to KYI.

Sreenivasan and Bershadskii (2006) proposed an approximation for $\Delta(r)$ based on an expansion of $\Delta(r)$ in powers of $\log(r/r_{\mathrm{m}})$, and argued that experimental and DNS data fit well to $r_{\mathrm{m}}/\eta \propto R_\lambda^\mu$ with $\mu = 0.73 \pm 0.05$. The difference between this and the exponents in (1.14) and (1.16) is not surprising in view of the fact that the data set used for fitting includes data both from decaying turbulent flows and from forced turbulent flows.

1.3.3 Spectral space

The NS equation for homeogeneous and isotropic (HI) turbulence gives the spectral equation,

$$\frac{\partial}{\partial t} E(k) = T_E(k) - 2\nu k^2 E(k) + \hat{F}(k), \tag{1.17}$$

where $E(k)$ is the energy spectrum, $T_E(k)$ is the energy transfer function due to the nonlinear interaction, and $\hat{F}(k)$ represents the energy input due to the forcing $\mathbf{f}$.

Integrating (1.17) with respect to k from K to ∞ and replacing K with k gives

$$\Pi(k) = \langle \epsilon \rangle - 2\nu \int_0^k p^2 E(p)dp - \int_k^\infty \hat{F}(p)dp + \int_k^\infty \frac{\partial}{\partial t} E(p)dp, \tag{1.18}$$

where $\Pi(k)$ is the energy flux across the wavenumber k, and is defined by

$$\Pi(k) \equiv \int_0^k T_E(p)dp,$$

where we have used

$$\int_0^\infty T_E(k)dk = 0, \qquad \langle \epsilon \rangle = 2\nu \int_0^\infty k^2 E(k)dk.$$