

Contents

<i>Preface</i>	<i>page</i> xiii
0 Introductory remarks	1
0.1 Why p -adic differential equations?	1
0.2 Zeta functions of varieties	3
0.3 Zeta functions and p -adic differential equations	5
0.4 A word of caution	7
Notes	8
Exercises	9
Part I Tools of p-adic Analysis	11
1 Norms on algebraic structures	13
1.1 Norms on abelian groups	13
1.2 Valuations and nonarchimedean norms	16
1.3 Norms on modules	17
1.4 Examples of nonarchimedean norms	25
1.5 Spherical completeness	28
Notes	31
Exercises	33
2 Newton polygons	35
2.1 Introduction to Newton polygons	35
2.2 Slope factorizations and a master factorization theorem	38
2.3 Applications to nonarchimedean field theory	41
Notes	42
Exercises	43
3 Ramification theory	45
3.1 Defect	46
3.2 Unramified extensions	47
	v

vi	<i>Contents</i>	
3.3	Tamely ramified extensions	49
3.4	The case of local fields	52
	Notes	53
	Exercises	54
4	Matrix analysis	55
4.1	Singular values and eigenvalues (archimedean case)	56
4.2	Perturbations (archimedean case)	60
4.3	Singular values and eigenvalues (nonarchimedean case)	62
4.4	Perturbations (nonarchimedean case)	68
4.5	Horn's inequalities	71
	Notes	72
	Exercises	74
	Part II Differential Algebra	75
5	Formalism of differential algebra	77
5.1	Differential rings and differential modules	77
5.2	Differential modules and differential systems	80
5.3	Operations on differential modules	81
5.4	Cyclic vectors	84
5.5	Differential polynomials	85
5.6	Differential equations	87
5.7	Cyclic vectors: a mixed blessing	87
5.8	Taylor series	90
	Notes	91
	Exercises	91
6	Metric properties of differential modules	93
6.1	Spectral radii of bounded endomorphisms	93
6.2	Spectral radii of differential operators	95
6.3	A coordinate-free approach	102
6.4	Newton polygons for twisted polynomials	104
6.5	Twisted polynomials and spectral radii	105
6.6	The visible decomposition theorem	107
6.7	Matrices and the visible spectrum	109
6.8	A refined visible decomposition theorem	112
6.9	Changing the constant field	114
	Notes	116
	Exercises	117
7	Regular singularities	118
7.1	Irregularity	119

<i>Contents</i>		vii
7.2	Exponents in the complex analytic setting	120
7.3	Formal solutions of regular differential equations	123
7.4	Index and irregularity	126
7.5	The Turrittin–Levelt–Hukuhara decomposition theorem	127
	Notes	129
	Exercises	130
 Part III p-adic Differential Equations on Discs and Annuli		 133
8	Rings of functions on discs and annuli	135
8.1	Power series on closed discs and annuli	136
8.2	Gauss norms and Newton polygons	138
8.3	Factorization results	140
8.4	Open discs and annuli	143
8.5	Analytic elements	144
8.6	More approximation arguments	147
	Notes	149
	Exercises	150
9	Radius and generic radius of convergence	151
9.1	Differential modules have no torsion	152
9.2	Antidifferentiation	153
9.3	Radius of convergence on a disc	154
9.4	Generic radius of convergence	155
9.5	Some examples in rank 1	157
9.6	Transfer theorems	158
9.7	Geometric interpretation	160
9.8	Subsidiary radii	162
9.9	Another example in rank 1	162
9.10	Comparison with the coordinate-free definition	164
	Notes	165
	Exercises	166
10	Frobenius pullback and pushforward	168
10.1	Why Frobenius descent?	168
10.2	p th powers and roots	169
10.3	Frobenius pullback and pushforward operations	170
10.4	Frobenius antecedents	172
10.5	Frobenius descendants and subsidiary radii	174
10.6	Decomposition by spectral radius	176
10.7	Integrality of the generic radius	180
10.8	Off-center Frobenius antecedents and descendants	181

Notes	182
Exercises	183
11 Variation of generic and subsidiary radii	184
11.1 Harmonicity of the valuation function	185
11.2 Variation of Newton polygons	186
11.3 Variation of subsidiary radii: statements	189
11.4 Convexity for the generic radius	190
11.5 Measuring small radii	191
11.6 Larger radii	193
11.7 Monotonicity	195
11.8 Radius versus generic radius	197
11.9 Subsidiary radii as radii of optimal convergence	198
Notes	199
Exercises	200
12 Decomposition by subsidiary radii	201
12.1 Metrical detection of units	202
12.2 Decomposition over a closed disc	203
12.3 Decomposition over a closed annulus	207
12.4 Decomposition over an open disc or annulus	209
12.5 Partial decomposition over a closed disc or annulus	210
12.6 Modules solvable at a boundary	211
12.7 Solvable modules of rank 1	212
12.8 Clean modules	214
Notes	216
Exercises	216
13 p-adic exponents	218
13.1 p -adic Liouville numbers	218
13.2 p -adic regular singularities	221
13.3 The Robba condition	222
13.4 Abstract p -adic exponents	223
13.5 Exponents for annuli	225
13.6 The p -adic Fuchs theorem for annuli	231
13.7 Transfer to a regular singularity	234
Notes	237
Exercises	238
Part IV Difference Algebra and Frobenius Modules	241
14 Formalism of difference algebra	243
14.1 Difference algebra	243

<i>Contents</i>	ix
14.2 Twisted polynomials	246
14.3 Difference-closed fields	247
14.4 Difference algebra over a complete field	248
14.5 Hodge and Newton polygons	254
14.6 The Dieudonné–Manin classification theorem	256
Notes	258
Exercises	260
15 Frobenius modules	262
15.1 A multitude of rings	262
15.2 Frobenius lifts	264
15.3 Generic versus special Frobenius lifts	266
15.4 A reverse filtration	269
Notes	271
Exercises	272
16 Frobenius modules over the Robba ring	273
16.1 Frobenius modules on open discs	273
16.2 More on the Robba ring	275
16.3 Pure difference modules	277
16.4 The slope filtration theorem	279
16.5 Proof of the slope filtration theorem	281
Notes	284
Exercises	286
Part V Frobenius Structures	289
17 Frobenius structures on differential modules	291
17.1 Frobenius structures	291
17.2 Frobenius structures and the generic radius of convergence	294
17.3 Independence from the Frobenius lift	296
17.4 Slope filtrations and differential structures	298
17.5 Extension of Frobenius structures	298
Notes	299
Exercises	300
18 Effective convergence bounds	301
18.1 A first bound	301
18.2 Effective bounds for solvable modules	302
18.3 Better bounds using Frobenius structures	306
18.4 Logarithmic growth	308
18.5 Nonzero exponents	310

Notes	310
Exercises	311
19 Galois representations and differential modules	313
19.1 Representations and differential modules	314
19.2 Finite representations and overconvergent differential modules	316
19.3 The unit-root p -adic local monodromy theorem	318
19.4 Ramification and differential slopes	321
Notes	323
Exercises	325
20 The p-adic local monodromy theorem	326
20.1 Statement of the theorem	326
20.2 An example	328
20.3 Descent of sections	329
20.4 Local duality	332
20.5 When the residue field is imperfect	333
Notes	335
Exercises	337
21 The p-adic local monodromy theorem: proof	338
21.1 Running hypotheses	338
21.2 Modules of differential slope 0	339
21.3 Modules of rank 1	341
21.4 Modules of rank prime to p	342
21.5 The general case	343
Notes	343
Exercises	344
Part VI Areas of Application	345
22 Picard–Fuchs modules	347
22.1 Origin of Picard–Fuchs modules	347
22.2 Frobenius structures on Picard–Fuchs modules	348
22.3 Relationship to zeta functions	349
Notes	350
23 Rigid cohomology	352
23.1 Isocrystals on the affine line	352
23.2 Crystalline and rigid cohomology	353
23.3 Machine computations	354
Notes	355

	<i>Contents</i>	xi
24	<i>p</i>-adic Hodge theory	357
24.1	A few rings	357
24.2	(ϕ, Γ) -modules	359
24.3	Galois cohomology	361
24.4	Differential equations from (ϕ, Γ) -modules	362
24.5	Beyond Galois representations	363
	Notes	364
	<i>References</i>	365
	<i>Notation</i>	374
	<i>Index</i>	376