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p-adic Differential Equations

Over the last 50 years the theory of p-adic differential equations has grown into an active area of research in its own right, and has important applications to number theory and to computer science. This book, the first comprehensive and unified introduction to the subject, improves and simplifies existing results as well as including original material.

Based on a course given by the author at MIT, this modern treatment is accessible to graduate students and researchers. Exercises are included at the end of each chapter to help the reader review the material, and the author also provides detailed references to the literature to aid further study.

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Preface

This book is an outgrowth of a course, taught by the author at MIT during fall 2007, on p-adic ordinary differential equations. The target audience was graduate students with some prior background in algebraic number theory, including exposure to p-adic numbers, but not necessarily with any background in p-adic analytic geometry (of either the Tate or Berkovich flavors).

Custom would dictate that ordinarily this preface would continue with an explanation of what p-adic differential equations are, and why they matter. Since we have included a whole chapter on this topic (Chapter 0), we will devote this preface instead to a discussion of the origin of the book, its general structure, and what makes it different from previous books on the subject.

The subject of p-adic differential equations has been treated in several previous books. Two that we used in preparing the MIT course, and to which we make frequent reference in the text, are those of Dwork, Gerotto, and Sullivan [**80**] and of Christol [**42**]. Another existing book is that of Dwork [**78**], but it is not a general treatise; rather, it focuses in detail on hypergeometric functions.

However, this book develops the theory of p-adic differential equations in a manner that differs significantly from most prior literature. Key differences include the following.

- We limit our use of cyclic vectors. This requires an initial investment in the study of matrix inequalities (Chapter 4) and lattice approximation arguments (especially Lemma 8.6.1), but it pays off in significantly stronger results.
- We introduce the notion of a Frobenius descendant (Chapter 10). This complements the older construction of Frobenius antecedents, particularly in dealing with certain boundary cases where the antecedent method does not apply.

As a result, we end up with some improvements of existing results, including the following. (Some of these can also be found in an upcoming book of Christol [**46**], whose development we learned about only after this book was mostly complete.)

- We refine the Frobenius antecedent theorem of Christol and Dwork (Theorem 10.4.2).
- We extend some results of Christol and Dwork, on the variation of the generic radius of convergence, to subsidiary radii (Theorem 11.3.2).
- We extend Young's geometric interpretation of subsidiary generic radii of convergence beyond the range of applicability of Newton polygons (Theorem 11.9.2).
- We give quantitative versions of the Christol–Mebkhout decomposition theorem for differential modules on an annulus that are applicable even when the modules are not solvable at a boundary (Theorems 12.2.2 and 12.3.1).
- We give a somewhat simplified treatment of the theory of p-adic exponents (Theorems 13.5.5, 13.5.6, and 13.6.1).
- We sharpen the bound in the Christol transfer theorem to a disc containing a regular singularity with exponents in $\mathbb{Z}_p$ (Theorem 13.7.1).
- We give a general version of the Dieudonné–Manin classification theorem for difference modules over a complete nonarchimedean field (Theorem 14.6.3).
- We give improvements on the Christol–Dwork–Robba effective bounds for solutions of p-adic differential equations (Theorems 18.2.1 and 18.5.1) and some related bounds that apply in the presence of a Frobenius structure (Theorem 18.3.3). The latter can be used to recover a theorem of Chiarellotto and Tsuzuki concerning the logarithmic growth of solutions of differential equations with Frobenius structure (Theorem 18.4.5).
- We state a relative version of the p-adic local monodromy theorem, formerly Crew's conjecture (Theorem 20.1.4), and describe in detail how it may be derived either from the p-adic index theory of Christol and Mebkhout, which we treat in detail in Chapter 13, or from the slope theory for Frobenius modules of Kedlaya, which we only sketch, in Chapter 16.

Some of the new results are relevant in theory (in the study of higher-dimensional p-adic differential equations, largely in the context of the *semistable reduction problem* for overconvergent F-isocrystals, for which see [**138**] and [**143**]) or in practice (in the explicit computation of solutions of p-adic differential equations, e.g., for the machine computation of zeta functions of particular varieties, for which see [**139**]). There is also some relevance, entirely outside number theory, to the study of flat connections on complex analytic varieties (see [**144**]).

Although some applications involve higher-dimensional p-adic analytic spaces, this book treats exclusively p-adic *ordinary* differential equations. In joint work with Liang Xiao [**145**], we have developed some extensions to higher-dimensional spaces.

Each individual chapter of this book exhibits the following basic structure. Before the body of the chapter, we give a brief introduction explaining what is to be discussed and often setting some running notations or hypotheses. After the body of the chapter, we typically include a section of afternotes, in which we provide detailed references for results in that chapter, fill in historical details, and add additional comments. (This practice is modeled on that in [**94**], although we do not carry it out quite as fully.) Note that we have a habit of attributing to various authors slightly stronger versions of their theorems than the ones they originally stated; to avoid complicating the discussion in the text, we resolve these misattributions in the afternotes instead. At the end of a chapter we typically include a few exercises; a fair number of these request proofs of results which are stated and used in the text but whose proofs pose no unusual difficulties.

The chapters themselves are grouped into several parts, which we now describe briefly. (Chapter 0, being introductory, does not fit into this grouping.)

Part I is preliminary, collecting some basic tools of p-adic analysis. However, it also includes some facts of matrix analysis (the study of the variation of numerical invariants attached to matrices as a function of the matrix entries) which may not be familiar to the typical reader.

Part II introduces some formalism of differential algebra, such as differential rings and modules, twisted polynomials, and cyclic vectors, and applies these to fields equipped with a nonarchimedean norm.

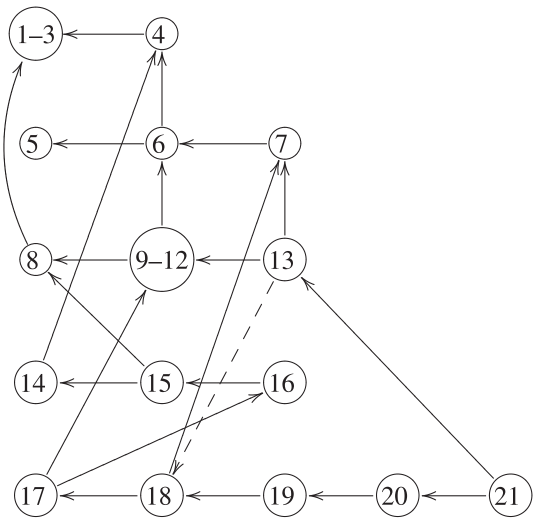
Part III begins the study of p-adic differential equations in earnest, developing some basic theory for differential modules on rings and annuli, including the Christol–Dwork theory of variation of the generic radius of convergence and the Christol–Mebkhout decomposition theory. We also include a treatment of p-adic exponents, culminating in the Christol–Mebkhout structure theorem for p-adic differential modules on an annulus satisfying the Robba condition (i.e., having intrinsic generic radius of convergence everywhere equal to 1).

Part IV introduces some formalism of difference algebra, and presents (without full proofs) the theory of slope filtrations for Frobenius modules over the Robba ring.

Part V introduces the concept of a Frobenius structure on a p-adic differential module, to the point of stating the p-adic local monodromy theorem and sketching briefly the proof techniques using either p-adic exponents or Frobenius slope filtrations. We also discuss effective convergence bounds for solutions of p-adic differential equations.

Part VI consists of a series of brief discussions of several areas of application of the theory of p-adic differential equations. These are somewhat more didactic, and much less formal, than in the other parts; they are meant primarily as suggestions for further reading.

The following diagram indicates the logical dependencies of the chapters. To keep the diagram manageable, we have grouped together some chapters (1–3 and 9–12) and omitted Chapter 0 and the chapters of Part VI. The reader should be aware that there is one forward reference, from Chapter 13 to Chapter 18, but the graph remains acyclic. (There are some additional forward references between Chapters 1 and 2, but these should not cause any difficulty.)



As noted above we have not assumed that the reader is familiar with rigid analytic geometry and so have phrased all statements more concretely in terms of rings and modules. Although we expect that the typical reader has at least a passing familiarity with p-adic numbers, for completeness we include a rapid development of the algebra of complete rings and fields in the first few chapters of the book. This development, when read on its own, may appear somewhat idiosyncratic; its design is justified by the reuse of some material in later chapters.

We would like to think that the background needed is that of a two-semester undergraduate abstract algebra course. However, some basic notions from commutative algebra do occasionally intervene, including flat modules, exact sequences, and the snake lemma. It may be helpful to have a well-indexed text on commutative algebra within arm's reach; we like Eisenbud's book [**84**], but the far slimmer Atiyah and Macdonald [**9**] should also suffice.

The author would like to thank the participants of the MIT course 18.787 ("Topics in number theory", fall 2007) for numerous comments on the lecture notes which ultimately became this book. Particular thanks are due to Ben Brubaker and David Speyer for giving guest lectures, and to Chris Davis, Hansheng Diao, David Harvey, Raju Krishnamoorthy, Ruochuan Liu, Eric Rosen, and especially Liang Xiao for providing feedback. Additional feedback was provided by Francesco Baldassarri, Laurent Berger, Bruno Chiarellotto, Gilles Christol, Ricardo García López, Tim Gowers, and Andrea Pulita.

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