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Ergodic Control of Diffusion Processes

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Dedicated to our parents:

Theodore and Helen,

Shripad and Sarita,

Sudhir Kumar and Uma.

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Preface

Ergodic is a term appropriated from physics that derives from the Greek words *έργον* and *οδός*, meaning "work" and "path." In the context of controlled Markov processes it refers to the problem of minimizing a time averaged penalty, or cost, over an infinite time horizon. It is of interest in situations when transients are fast and therefore relatively unimportant, and one is essentially comparing various possible equilibrium behaviors. One typical situation is in communication networks, where continuous time and space models arise as scaled limits of the underlying discrete state and/or time phenomena.

Ergodic cost differs from the simpler "integral" costs such as finite horizon or infinite horizon discounted costs in several crucial ways. Most importantly, one is looking at a cost averaged over infinite time, whence any finite initial segment is irrelevant as it does not affect the cost. This counterintuitive situation is also the reason for the fundamental difficulty in handling this problem analytically – one cannot use for this problem the naive dynamic programming heuristic because it is perforce based on splitting the time horizon into an initial segment and the rest. One is thus obliged to devise altogether different techniques to handle the ergodic cost. One of them, the more familiar one, is to treat it as a limiting case of the infinite horizon discounted cost control problem as the discount factor tends to zero. This "vanishing discount" approach leads to the correct dynamic programming, or "Hamilton–Jacobi–Bellman" (HJB) equation for the problem, allowing one to characterize optimal control policies at least in the "nicer" situations when convenient technical hypotheses hold. It also forms the basis of the approach one takes in order to do what one can in cases when these hypotheses do not hold. Dynamic programming, though the most popular "classical" approach to control problems, is not the only one. An alternative approach that is gaining currency, particularly as it allows one to handle some nonclassical variants and because of the numerical schemes it facilitates, is that based on casting the control as an infinite dimensional convex (in fact, *linear*) program. Our treatment of ergodic control straddles both lines of thought, often combining them to advantage.

Historically, the theory of ergodic control was first developed for discrete time and discrete state space Markov chains (see Arapostathis *et al.* [6] for an extensive survey and historical account). The results therein, as always, are suggestive of what one might expect in the continuous time continuous state space situation. That in itself, however, is of little help, as the technical difficulties in carrying out the program are highly nontrivial. Not surprisingly, the theory in the latter case has been slow to follow, some of it being of a fairly recent vintage. This book gives a comprehensive, integrated treatment of these developments. It begins with the better understood "nondegenerate, complete observations" case, and leads to the more difficult issues that arise when either the nondegeneracy assumption or the assumption of complete observations fail.

Our focus is primarily on controlled *diffusion processes*, a special class of Markov processes in continuous time and space. We build the basic theory of such processes in Chapter 2, following a quick review of the relevant aspects of ergodic theory of Markov processes in Chapter 1. This itself will serve as a comprehensive account of the probabilistic theory of controlled diffusions. It is an update on an earlier monograph [28], and appears in one place at this level of generality and extent for the first time. It forms the backdrop for the developments in the rest of this monograph and will also serve as useful source material for stochastic control researchers working on other problems. Chapter 3 gives a complete account of the relatively better understood case of controlled diffusions when the state is observed and the diffusion matrix is nondegenerate. The latter intuitively means that the noise enters all components of the state space evenly. The smoothing properties of the noise then yield sufficient regularity of various averages of interest. This permits us to use to our advantage the theory of nondegenerate second order elliptic partial differential equations. Our pursuits here are typical of all control theory: existence of optimal controls and necessary and sufficient conditions for optimality. We employ the infinite dimensional convex (in fact, linear) programming perspective for the former and the vanishing discount paradigm for the latter. The theory here is rich enough that it allows us to be a bit more ambitious and handle some nonclassical classes of problems as well with some additional work. One of these, that of switching diffusions, is treated in Chapter 5. This incorporates "regime-switching" phenomena and involves some nontrivial extensions of the theory of elliptic PDEs used in Chapter 3, to systems of elliptic PDEs. Chapter 4, in turn, studies several other spin-offs: the first is constrained and multi-objective problems and is followed by singularly perturbed ergodic control problems involving two time-scales, the aim being to justify a lower dimensional approximation obtained by averaging the slow dynamics over the equilibrium behavior of the fast components. It also studies diffusions in a bounded domain, and works out an example in detail.

With Chapter 6, we enter the vastly more difficult terrain of degeneracy. This chapter in particular is devoted to developing the abstract framework of ergodic control of martingale problems that will form the backdrop of subsequent chapters. A key development in this chapter is an important generalization of Echeverria's celebrated criterion for stationary distributions to controlled martingale problems on a Polish space. The rest of the chapter is devoted to characterizing extremal solutions, the abstract linear programming formulation, and the existence theorems for optimal solutions in various classes of interest. These results are specialized in Chapter 7 to controlled diffusions with degeneracy, where they lead to existence theorems for optimal controls with little extra effort. For two special cases, viz., the so-called "asymptotically flat" diffusions and "partially degenerate" diffusions, we also derive dynamic programming principles. This takes the form of an HJB equation, albeit interpreted in the viscosity sense. Chapter 8 considers nondegenerate diffusions under partial observations. The standard approach, summarized in this chapter, is to reduce the problem to a completely observed case by moving over to the "sufficient statistics" such as the regular conditional law of the state given observed quantities till the time instant in question. The evolution of this law is given by the stochastic PDE describing the associated measure-valued nonlinear filter, thus making it a completely observed but infinite dimensional and in a certain sense, degenerate problem. This formulation, however, does allow us to apply the theory of Chapter 6 and develop the existence theory for optimal controls, as well as to derive a martingale dynamic programming principle under suitable hypotheses.

The epilogue in Chapter 9, an important component of this work, sketches several open issues. This being a difficult problem area has its advantages as well – there is still much work left to be done, some of it crying out for novel approaches and proof techniques. We hope that this book will spur other researchers to take up some of these issues and see successful resolution of at least some of them in the near future.

We extend our heartfelt thanks to our numerous friends, colleagues and students, far too numerous to list, who have helped in this project in a variety of ways. Special thanks are due to Professor S.R.S. Varadhan, who, in addition to his pervasive influence on the field as a whole, also made specific technical recommendations in the early stages of our work in this area, which had a critical impact on the way it developed.

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It has been both a challenge and a pleasure for us to write this book. We particularly cherish the time we spent together in Mumbai working on it together, the hectic schedule punctuated by the culinary delights of south Mumbai restaurants and cafes.

Frequently Used Notation

The sets of real numbers, integers and natural numbers are denoted by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ respectively. Also, $\mathbb{R}_+$ ($\mathbb{Z}_+$) denotes the set of nonnegative real numbers (nonnegative integers). We use ":=" to mean "is defined as," and "≡" to mean "identically equal to." If f and g are real-valued functions (or real numbers), we define

$$f \wedge g := \min\{f, g\}\,, \qquad f \vee g := \max\{f, g\}\,,$$
$$f^+ := f \vee 0\,, \qquad f^- := (-f) \vee 0\,.$$

For a subset A of a topological space, ∂A denotes its boundary, A^c its complement and $\bar{A}$ its closure. The indicator function of A is denoted by $\mathbb{I}_A$. In the interest of readability for a set that is explicitly defined by an expression $\{\cdot\}$ we denote its indicator function as $\mathbb{I}\{\cdot\}$.

The Borel σ-field of a topological space E is denoted by $\mathscr{B}(E)$. Metric spaces are in general viewed as equipped with their Borel σ-field, and therefore the notation $\mathscr{P}(E)$ for the set of probability measures on $\mathscr{B}(E)$ of a metric space E is unambiguous. The set of bounded real-valued measurable functions on a metric space E is denoted by $\mathcal{B}(E)$. The symbol $\mathbb{E}$ always denotes the expectation operator, and $\mathbb{P}$ the probability.

The standard Euclidean norm is denoted as $|\cdot|$. The term *domain* in $\mathbb{R}^d$ refers to a non-empty open connected subset of the Euclidean space $\mathbb{R}^d$. If D and D' are domains in $\mathbb{R}^d$, we use the notation $D \Subset D'$ to indicate that $\bar{D} \subset D'$. Also $|D|$ stands for the Lebesgue measure of a bounded domain D. We introduce the following notation for spaces of real-valued functions on a domain $D \subset \mathbb{R}^d$. The space $L^p(D)$, for $p \in [1, \infty)$, stands for the Banach space of (equivalence classes) of measurable functions f satisfying $\int_D |f(x)|^p \,\mathrm{d}x < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in D. The space $C^k(D)$ ($C^\infty(D)$) refers to the class of all functions whose partial derivatives up to order k (of any order) exist and are continuous, $C^k_c(D)$ is the space of functions in $C^k(D)$ with compact support, and $C^k_0(\mathbb{R}^d)$ ($C^k_b(\mathbb{R}^d)$) is the subspace of $C^k(\mathbb{R}^d)$ consisting of those functions whose derivatives from order 0 to k vanish at infinity (are bounded). Also, the space $C^{k,r}(D)$ is the class of all functions whose partial derivatives up to order k are Hölder continuous of order r. Therefore $C^{0,1}(D)$ is precisely the space of Lipschitz continuous functions on D.

We adopt the notation $\partial_i := \frac{\partial}{\partial x_i}$, $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$, and $\partial_t := \frac{\partial}{\partial t}$. Also we adopt the standard summation rule for repeated indices, i.e., repeated subscripts and superscripts are summed from 1 through d. For example,

$$a^{ij}\partial_{ij}\varphi + b^i\partial_i\varphi := \sum_{i,j=1}^{d} a^{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial \varphi}{\partial x_i}\,.$$

The standard Sobolev space of functions on D whose generalized derivatives up to order k are in $L^p(D)$, equipped with its natural norm, is denoted by $\mathscr{W}^{k,p}(D)$, $k \geq 0$, $p \geq 1$. The closure of $C_c^\infty(D)$ in $\mathscr{W}^{k,p}(D)$ is denoted by $\mathscr{W}_0^{k,p}(D)$. If B is an open ball, then $\mathscr{W}_0^{k,p}(B)$ consists of all functions in $\mathscr{W}^{k,p}(B)$ which, when extended by zero outside B, belong to $\mathscr{W}^{k,p}(\mathbb{R}^d)$.

In general if $\mathcal{X}$ is a space of real-valued functions on D, $\mathcal{X}_{\mathrm{loc}}$ consists of all functions f such that $\varphi f \in \mathcal{X}$ for every $\varphi \in C_c^\infty(D)$. In this manner we obtain the spaces $L^p_{\mathrm{loc}}(D)$ and $\mathscr{W}^{2,p}_{\mathrm{loc}}(D)$.