# 1
# Preliminaries

We begin by reviewing some simple concepts regarding set systems, graphs, metric spaces, and computational complexity which will be used throughout this book. For more information on these topics see, for example, [48, 49, 79].

## 1.1 Sets, set systems, and partially ordered sets

In this section, we introduce useful terminology regarding sets, set systems, and partially ordered sets.

A finite set $V$ of cardinality $n$ will also be called an *n-set* and the $n$-set $\{1, 2, \ldots, n\}$ will be denoted by $\langle n \rangle$. A *set system* (over $V$) is a subset $\mathcal{V}$ of the *power set* $\mathbb{P}(V)$ of $V$, i.e., the set consisting of *all* subsets of $V$. The subsets in $\mathcal{V}$ are often also called the *clusters* in $\mathcal{V}$. For any non-negative integer $k \in \mathbb{N}_{\geq 0}$, the set system consisting of all $k$-subsets of $V$ will also be denoted by $\binom{V}{k}$, and the set system consisting of all subsets of $V$ of cardinality at least/at most $k$ will also be denoted by $\mathbb{P}_{\geq k}(V)$ or $\mathbb{P}_{\leq k}(V)$, respectively. Given a subset $A$ of a set $V$ and an element $x \in V$, we denote the union $A \cup \{x\}$ also by $A + x$ and the difference $A \setminus \{x\} = \{a \in A : a \neq x\}$ also by $A - x$. Also, given two subsets $A, B$ of $V$, we may write $A - B$ for $A \setminus B$.

Set systems are special instances of *partially ordered sets*, i.e., sets $U$ together with a binary relation "$\preceq$" defined on $U$ such that

$$u_1 \preceq u_2 \quad \text{and} \quad u_2 \preceq u_3 \Rightarrow u_1 \preceq u_3$$

and

$$u_1 \preceq u_2 \quad \text{and} \quad u_2 \preceq u_1 \iff u_1 = u_2$$

holds for all $u_1, u_2, u_3 \in U$ in which case the binary relation "$\preceq$" — or as well the (also transitive!) binary relation "$\prec$" defined by "$u \prec u' \iff u \preceq u'$ and $u \neq u'$" — is called a *partial order*. For any partially ordered set $U$, we denote by

$$\max(U) = \max_{\preceq}(U) := \{u \in U : \forall_{u' \in U}\ u \preceq u' \Rightarrow u = u'\}$$

the set of maximal elements in $U$ (relative to the partial order "$\preceq$") and by

$$\min(U) = \min_{\preceq}(U) := \{u \in U : \forall_{u' \in U}\ u' \preceq u \Rightarrow u = u'\}$$

the set of minimal elements in $U$ (relative to the partial order "$\preceq$"), we denote, for any $u \in U$, by $U_{\preceq u}$ the set of all $u' \in U$ with $u' \preceq u$ and by $U_{\prec u}$ the set of all $u' \in U_{\preceq u}$ that are distinct from $u$. We also consider any subset $U'$ of a partially ordered set $U = (U, \preceq)$ as being itself a partially ordered set relative to the restriction of the binary relation $\preceq$ to $U'$ which we keep denoting by $\preceq$ as long as no misunderstanding can arise. In particular, we denote by $U'_{\preceq u}$ the set $U_{\preceq u} \cap U'$ and by $U'_{\prec u}$ the set $U_{\prec u} \cap U'$. Furthermore, the elements in $\max(U_{\prec u})$ will sometimes also be called the *children* of $u$, and we will therefore denote the set $\max(U_{\prec u})$ also by $\mathrm{chld}_U(u)$.

In particular, we denote, for any set system $\mathcal{V} \subseteq \mathbb{P}(V)$ over a set $V$ and any subset $L$ of $V$, by $\mathcal{V}_{\subseteq L}$ the set of all $U \in \mathcal{V}$ with $U \subseteq L$ and by $\mathcal{V}_{\subset L}$ the set of all $U \in \mathcal{V}_{\subseteq L}$ with $U \subsetneq L$. We will also denote by $\bigcup \mathcal{V}$ the union $\bigcup_{U \in \mathcal{V}} U$ of all clusters in a set system $\mathcal{V}$ and by $\bigcap \mathcal{V}$ the intersection $\bigcap_{U \in \mathcal{V}} U$ of all clusters in $\mathcal{V}$.

Of particular significance will be partitions and hierarchies. A set system $\mathcal{V} \subseteq \mathbb{P}(V)$ is defined to be a *partition* if it is contained in the set $\mathbb{P}_{\geq 1}(V) := \{U \subseteq V : U \neq \emptyset\}$ consisting of all non-empty subsets of $V$ and $U_1 \cap U_2 = \emptyset$ holds for any two distinct clusters $U_1, U_2$ in $\mathcal{V}$, it is called a *partition of* $V$ if, in addition, $\bigcup \mathcal{V} = V$ holds, it is called a *bipartition* or a *split* (of $V$) if it is a partition (of $V$) and contains exactly two distinct clusters, and every cluster in a partition will also be called a *part* of that partition. Often, we will also refer to splits by the letter $S$ and denote a split $S$ of the form $\{A, B\}$ by $A|B$. We will not distinguish between $A|B$ and $B|A$ as both terms stand for the same split $\{A, B\}$. Given a split $S = A|B$, the number $\min\{|A|, |B|\}$ will also be called its *size* and denoted by $\|S\|$ or, as well, by $\|A|B\|$. A split of size 1 is also called *trivial*, and a split of size $k$ a *$k$-split*. And, given an element $x \in X$ and a split $S = A|B$ with $x \in A \cup B$, we denote that subset, $A$ or $B$, in $S$ that contains the element $x$ by $S(x)$ and its complement in $A \cup B$ by $\overline{S}(x)$.

Clearly, $\bigcup \mathcal{U}_1 \cup \bigcup \mathcal{U}_2 = \bigcup(\mathcal{U}_1 \cup \mathcal{U}_2)$ holds for any two subcollections $\mathcal{U}_1, \mathcal{U}_2$ of a set system $\mathcal{V} \subseteq \mathbb{P}(V)$ while a set system $\mathcal{V} \subseteq \mathbb{P}_{\geq 1}(V)$ is a partition if and only if $\bigcup \mathcal{U}_1 \cap \bigcup \mathcal{U}_2 = \bigcup(\mathcal{U}_1 \cap \mathcal{U}_2)$ holds for any two subcollections $\mathcal{U}_1, \mathcal{U}_2$ of $\mathcal{V}$.

Further, a set system $\mathcal{H} \subseteq \mathbb{P}(V)$ is defined to be a *hierarchy* (over $V$) if it is contained in $\mathbb{P}_{\geq 1}(V)$ and $H_1 \cap H_2 \in \{\emptyset, H_1, H_2\}$ holds for any two clusters $H_1, H_2 \in \mathcal{H}$. Clearly, $\text{chld}_{\mathcal{H}}(H)$ must be a partition for every $H$ in a hierarchy $\mathcal{H}$: Indeed, if $H_1$ and $H_2$ are two distinct children of some cluster $H$ in a hierarchy $\mathcal{H}$, we must have $H_1 \cap H_2 = \emptyset$ as neither $H_1 \cap H_2 = H_1$ nor $H_1 \cap H_2 = H_2$ can hold.

It is easy to see that, conversely, a set system $\mathcal{V} \subseteq \mathbb{P}_{\geq 1}(V)$ must be a hierarchy if $\text{chld}_{\mathcal{V}}(U)$ is a partition for every $U$ in $\mathcal{V}$ provided $V$ is finite and a member of $\mathcal{V}$: Indeed, if this holds and if $U_1$ and $U_2$ are any two clusters in $\mathcal{V}$, there exists — in view of $V \in \mathcal{V}$ — an inclusion-minimal cluster $U$ in $\mathcal{V}$ containing $U_1 \cup U_2$. If $U = U_1$ or $U = U_2$ holds, we have $U_1 \cap U_2 \in \{U_1, U_2\}$. Otherwise, there exist largest proper subsets $U_1', U_2'$ of $U$ in $\mathcal{V}$ that contain $U_1$ and $U_2$, respectively, and we must have $U_1' \neq U_2'$ by the choice of $U$ (as, otherwise, $U_1' = U_2'$ would be a smaller cluster than $U$ in $\mathcal{V}$ that contains $U_1 \cup U_2$). So, $U_1'$ and $U_2'$ must be distinct members of the partition $\text{chld}_{\mathcal{V}}(U)$ and, therefore, disjoint, implying that also $U_1 \cap U_2 \subseteq U_1' \cap U_2' = \emptyset$ must be empty. So, $U_1 \cap U_2 \in \{U_1, U_2, \emptyset\}$ must indeed hold for any two clusters $U_1$ and $U_2$ in $\mathcal{V}$.

It is also easy to see that every hierarchy $\mathcal{H}$ over an $n$-set contains at most $2n - 1$ clusters: Indeed, this clearly holds in case $n := 1$, and if it holds for any hierarchy over any proper subset of an $n$-set $V$, then it holds for $\mathcal{H}$, too, in view of $\mathcal{H} \subseteq V + \dot{\bigcup}_{H \in \text{chld}_{\mathcal{H}}(V)} \mathcal{H}_{\subseteq H}$ and, hence,

$$|\mathcal{H}| \leq 1 + \sum_{H \in \text{chld}_{\mathcal{H}}(V)} |\mathcal{H}_{\subseteq H}|,$$

the fact that $\sum_{U \in \mathcal{V}} |U| \leq n$ must hold for every partition $\mathcal{V} \subseteq \mathbb{P}(V)$, and that $\mathcal{H}_{\subseteq H}$ is a hierarchy over $H$ for every $H \in \mathcal{H}$. So,

$$|\mathcal{H}| \leq 1 + \sum_{H \in \text{chld}_{\mathcal{H}}(V)} (2\,|H| - 1) \leq 1 + 2n - |\text{chld}_{\mathcal{H}}(V)| \leq 2n - 1$$

must hold in case $2 \leq |\text{chld}_{\mathcal{H}}(V)|$. And it must hold in case $|\text{chld}_{\mathcal{H}}(V)| < 2$ as this implies that even $1 + \sum_{U \in \text{chld}_{\mathcal{H}}(V)} (2\,|U| - 1) \leq 1 + 2\,(n-1) - 1 = 2n - 2$ must hold.

In particular, we have $|\mathcal{H}| = 2n - 1$ if and only if $V \in \mathcal{H}$ holds, $\text{chld}_{\mathcal{H}}(V)$ is a split of $V$, and $|\mathcal{H}_{\subseteq U}| = 2\,|U| - 1$ holds for both clusters $U \in \text{chld}_{\mathcal{H}}(V)$ — so, by recursion, this holds if and only if $V \in \mathcal{H}$ holds and $\text{chld}_{\mathcal{H}}(U)$ is a split of $U$ for every cluster $U \in \mathcal{H}$ with $|U| \geq 2$.

More generally, the following fact is well known and easy to see:

**Lemma 1.1** *Given a hierarchy $\mathcal{H}$ over a finite set $V$ of cardinality $n$, the following assertions are equivalent:*

(i) $|\mathcal{H}| = 2n - 1$ *holds.*
(ii) $\mathcal{H}$ *contains* $V$ *and* $\mathrm{chld}_{\mathcal{H}}(H)$ *is a split of* $H$ *for every cluster* $H \in \mathcal{H}$ *with* $|H| \geq 2$.
(iii) $\mathcal{H}$ *is a maximal hierarchy over* $V$*, i.e.,* $U \in \mathcal{H}$ *holds for every subset* $U$ *of* $V$ *with* $U \cap H \in \{U, H, \emptyset\}$ *for every cluster* $H \in \mathcal{H}$.
(iv) $\mathcal{H}$ *contains* $V$ *and all one-element subsets of* $V$*, and* $H_2 - H_1 \in \mathcal{H}$ *holds for any two subsets* $H_1, H_2 \in \mathcal{H}$ *with* $H_1 \in \mathrm{chld}_{\mathcal{H}}(H_2)$.

*Proof* We have seen already that (i) $\iff$ (ii) holds. And it is also clear that (i) $\Rightarrow$ (iii) holds: Otherwise, there would exist a hierarchy over $V$ containing more than $2n-1$ clusters. And (iii) $\Rightarrow$ (iv) holds as $U \cap H \in \{U, H, \emptyset\}$ holds for every cluster $H \in \mathcal{H}$ for $U := V$ or $U$ a one-element subset of $V$. It also holds for $U := H_2 - H_1 \in \mathcal{H}$ in case $H_2 \in \mathcal{H}$ holds and $H_1$ is a child of $H_2$: Indeed, $U \subseteq H$ holds in case $H_2 \subseteq H$, $H \subseteq U$ holds in case $H \subseteq H_2$ and $H \cap H_1 = \emptyset$, and $H \cap U = \emptyset$ holds in case $H \cap H_2 = \emptyset$ and in case $H \subseteq H_1$. Finally, if neither $H_2 \subseteq H$ nor $H \cap H_1 = \emptyset$ nor $H \subseteq H_1$ holds, we would necessarily have $H \subsetneq H_2$ (in view of $H \cap H_2 \neq H_2, \emptyset$) and $H_1 \subsetneq H$ (in view of $H \cap H_1 \neq H, \emptyset$) in contradiction to our assumption that $H_1$ is a child of $H_2$ and that, therefore, $\{U \in \mathcal{H} : H_1 \subsetneq U \subsetneq H_2\} = \emptyset$ holds.

Finally, we have (iv) $\Rightarrow$ (ii) as $\mathrm{chld}_{\mathcal{H}}(H)$ must be a partition of $H$ for every $H \in \mathcal{H}$ whenever $\mathcal{H}$ contains all one-element subsets of $V$, and it must, of course, be a bipartition of $H$ if $H - H' \in \mathcal{H}$ holds for any $H' \in \mathrm{chld}_{\mathcal{H}}(H)$. □

Note that hierarchies over a set $V$ are sometimes required to also contain $V$ or the empty set or, as well, all one-element subsets of $V$ — see e.g., [28] where it was shown that a hierarchy $\mathcal{H}$ over an arbitrary set $V$ is a maximal hierarchy over $V$ if and only if $\mathcal{H}$ satisfies the condition (iv) and, in addition, $\bigcup \mathcal{C}, \bigcap \mathcal{C} \in \mathcal{H}$ holds for any "chain" $\mathcal{C}$ of clusters contained in $\mathcal{H}$ (i.e., any subset $\mathcal{C}$ of $\mathcal{H}$ with $C_1 \cap C_2 \in \{C_1, C_2\}$ for all $C_1, C_2 \in \mathcal{C}$) with $\bigcap \mathcal{C} \neq \emptyset$.

## 1.2 Graphs

A *graph* is a pair $G = (V, E)$ consisting of a non-empty set $V$, the *vertex set* of $G$, and a subset $E$ of $\binom{V}{2}$, the *edge set* of $G$. $G$ is called *finite* if its vertex

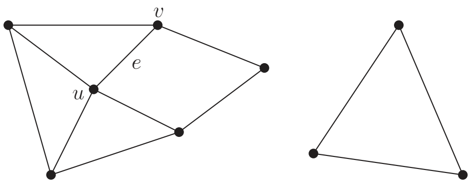


Figure 1.1 A (non-connected) graph with nine vertices and 12 edges.

set — and, hence, also its edge set — is finite. The elements of $V$ and $E$ are also called the *vertices* and the *edges* of $G$, respectively. Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic* if and only if there exists a bijective map $\iota : V \rightarrow V'$ with $\{u, v\} \in E \iff \{\iota(u), \iota(v)\} \in E'$ for all $u, v \in V$. Clearly, graphs can be viewed as particularly simple set systems, that is, set systems $\mathcal{V}$ for which any cluster $e \in \mathcal{V}$ has cardinality 2. In Figure 1.1, we present a (drawing of a) graph: Vertices are represented by dots, and edges by straight line segments.

Two vertices $u$ and $v$ of a graph $G$ are called *adjacent* if $\{u, v\}$ is an edge of $G$. For any edge $e = \{u, v\}$ of $G$, we call the vertices $u$ and $v$ the *endpoints* of $e$, and we will say that an edge $e \in E$ and a vertex $v \in V$ are *incident* if (and only if) $v \in e$ holds. The vertices that are adjacent to a vertex $v$ of $G$ are also called the *neighbors* of $v$ in $G$, the set of neighbors of $v$ in $G$ is denoted by $N_G(v)$ or just $N(v)$, and the set of edges that are incident to $v$ by $E_G(v)$ or just $E(v)$. The number of edges that are incident to a vertex $v$ — or, equivalently, the number of neighbors of $v$ — is called its *degree*, denoted by $\deg(v)$ or, more specifically, by $\deg_G(v)$.

For instance, referring to Figure 1.1, the vertex $u$ has degree 4 and is adjacent to the vertex $v$, and the edge $e$ is incident to both, $u$ and $v$.

A vertex of degree 1 is called a *leaf* (of $G$), and the unique edge $e$ of $G$ that is incident to a leaf $a$ is denoted by $e_G(a)$. Any such edge is also called a *pendant edge* while the unique vertex in $e_G(a)$ distinct from $a$ is denoted by $v_G(a)$.

Every vertex that is not a leaf is called an *interior vertex* of $G$, and every edge that is not a pendant edge is called an *interior edge*. We denote the set of interior vertices and edges of $G$ by $V_{int}(G)$ and $E_{int}(G)$, respectively. Clearly, "plucking off" all of the leaves and pendant edges from a graph $G = (V, E)$ yields a graph with vertex set $V_{int}(G)$ and edge set $E_{int}(G)$ that we dub the graph *derived* from $G$ and denote, for short, by $\partial G$.

A pair of distinct leaves $a, b$ is said to form a *cherry* (in $G$) — or, just, to be a cherry (of $G$) — if $v_G(a) = v_G(b)$ holds, i.e., if both leaves are adjacent

Cambridge University Press & Assessment
978-0-521-76832-0 — Basic Phylogenetic Combinatorics
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to the same vertex (which then must necessarily be an interior vertex, having degree at least 2). If $v$ has degree 3, the unique edge $e \in E(v)$ that is distinct from the two pendant edges $e_G(a)$ and $e_G(b)$ will be denoted by $e_G(a, b)$.

Frequently, we will refer to subgraphs of a given graph: A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ holds, and it is *the subgraph of G induced on $V'$ by G*, also denoted by $G[V']$, if — in addition — $E' = E_{V'} := E \cap \binom{V'}{2}$ holds, that is, if and only if $G'$ is the largest subgraph of $G$ with vertex set $V'$.

A *path* $\mathbf{p}$ in a graph $G = (V, E)$ is a sequence $v_0, v_1, \ldots, v_\ell$ of consecutively adjacent vertices of $G$, i.e., with $e_i := \{v_{i-1}, v_i\} \in E$ for all $i = 1, \ldots, \ell$, such that $v_{i-1} \neq v_{i+1}$ holds for all $i \in \{1, \ldots, \ell - 1\}$ — more specifically, any such sequence $v_0, v_1, \ldots, v_\ell$ will be called a *path of length* $\ell$ while the vertices $v_0, v_1, \ldots, v_\ell$ and the edges $e_1 = \{v_0, v_1\}, \ldots, e_\ell = \{v_{\ell-1}, v_\ell\}$ will be called the *vertices* and the *edges of* $\mathbf{p}$ or, as well, the vertices and edges that are *passed* by $\mathbf{p}$, and the sets $\{v_0, v_1, \ldots, v_\ell\}$ and $\{e_1, \ldots, e_\ell\}$ will also be denoted by $V(\mathbf{p})$ and $E(\mathbf{p})$, respectively. The vertex $v_0$ is also called the *starting point*, and the vertex $v_\ell$ the *end point* of $\mathbf{p}$ (though sometimes also both vertices, $v_0$ and $v_\ell$, may be referred to as its endpoints), and $\mathbf{p}$ is also called a path *from* $v_0$ *to* $v_\ell$.

A path $\mathbf{p}$ is called *proper* if all of its vertices except perhaps its starting and its end point are distinct, i.e., if $v_i \neq v_j$ holds for all $i, j \in \{0, 1, \ldots, \ell\}$ with $i \neq j$ and $\{i, j\} \neq \{0, \ell\}$, and it is called a (*cyclically*) *closed path* if its starting and its end point coincide, i.e., if $v_0 = v_\ell$ holds, its length is positive and, hence, exceeds 2, and also $v_1 \neq v_{\ell-1}$ holds.

In Figure 1.1, there is exactly one proper path of length 1, 2, 4, and 5, respectively, from $u$ to $v$, and two such paths of length 3.

A graph $G = (V, E)$ is *connected* if there exists, for any two vertices $u, v \in V$ of $G$, a path in $G$ with endpoints $u$ and $v$. More generally, a subset $U \subseteq V$ of the vertex set $V$ of a graph $G = (V, E)$ is *connected* (relative to $G$) if the associated induced subgraph $G[U]$ is connected. And a subset $F \subseteq E$ of the edge set $E$ of a graph $G = (V, E)$ is connected (relative to $G$) if the graph $\left(\bigcup F, F\right)$ is connected.

A *connected component* of a graph $G = (V, E)$ is an inclusion-maximal connected subset $U \subseteq V$ of $V$ or, equivalently, an inclusion-minimal nonempty subset $U$ of $V$ for which $e \subseteq U$ holds for all $e \in E$ with $e \cap U \neq \emptyset$. So, the graph in Figure 1.1, for example, "contains" exactly two distinct connected components.

Clearly, any two connected components of a graph $G$ either coincide or have an empty intersection. We denote the set of connected components of a graph

$G = (V, E)$ by $\pi_0(G)$, and we denote the (unique!) connected component of $G$ containing a given vertex $v \in V$ by $G(v)$.

It is also obvious that the set system $\pi_0(G)$ forms a partition of the vertex set $V$ of a graph $G = (V, E)$ and that, if $G$ is a connected graph with at least one interior vertex, the set $V_{int}(G)$ of its interior vertices is a connected subset of $G$ and the associated induced — and necessarily connected — subgraph $G[V_{int}(G)]$ coincides with the derived graph $\partial G$. More precisely, a graph $G$ with $V_{int}(G) \neq \emptyset$ is connected if and only if the derived graph $\partial G$ is connected and $G$ contains no *isolated vertices* or *isolated edges*, i.e., vertices or edges that form a connected component of $G$.

We will say that an edge $e \in E$ *separates* a vertex $v \in V$ from a vertex $u \in V$ if $G(v) = G(u)$ holds while the two connected components $G^{(e)}(u)$ and $G^{(e)}(v)$ of the graph $G^{(e)} := (V, E - e)$ containing $u$ and $v$, respectively, are distinct — that is, if there is a path in $G$ connecting $u$ and $v$, but every such path passes $e$. The set of all edges of $G$ separating the vertices $u$ and $v$ will be denoted by $E_G(u|v)$ or simply $E(u|v)$. And any edge $e = \{u, v\} \in E$ that separates its two endpoints $u$ and $v$ will be called a *bridge*.

More generally, we call a subset $E'$ of $E$ an *edge-cutset* of $G$ if $G(v)$ differs from $(V, E - E')(v)$ for at least one vertex $v \in V$. Analogously, a subset $U \subseteq V$ is a *vertex-cutset* of $G$ if there exist two vertices $u, u' \in V - U$ with $G(u) = G(u')$, but $G[V - U](u) \neq G[V - U](u')$. In particular, a vertex $v \in V$ such that $\{v\}$ is a vertex-cutset of $G$ is called a *cut vertex* of $G$.

A *cycle* is a finite connected graph all of whose vertices have degree 2. Clearly, a graph $G = (V, E)$ is a cycle if and only if it is finite and we can label its vertices as $v_1, v_2, \ldots, v_\ell$ ($\ell := |V|$) so that $E$ coincides with $\{\{v_1, v_2\}, \ldots, \{v_{\ell-1}, v_\ell\}, \{v_\ell, v_1\}\}$ in which case the sequence $v_0 := v_\ell, v_1, v_2, \ldots, v_\ell$ forms a proper closed path in $G$ that encompasses all vertices and edges of $G$. A *cycle in a graph* $G$ is a subgraph of $G$ that is a cycle. The graph in Figure 1.1 contains exactly four cycles of length 3 and 5, and three of length 4 and 6, respectively.

Clearly, an edge $e$ in a finite graph $G$ is contained in a cycle in $G$ if and only if it is contained in $\partial^k G$ for every natural number $k \in \mathbb{N}_{\geq 0}$ (where $\partial^k G$ is, of course, defined recursively by $\partial^0 G := G$ and $\partial^{k+1} G := \partial(\partial^k G)$ for every $k \in \mathbb{N}_{\geq 0}$).

A graph $T = (V, E)$ is a *tree* if it is connected and contains no cycles or equivalently, as every "shortest" closed path "is" a cycle, no closed path. A subgraph $T' = (V', E')$ of a tree $T$ that is connected is called a *subtree* of $T$ in which case it must coincide with the induced subgraph $T[V']$ of $T$ with vertex set $V'$. An example of a tree is given in Figure 1.2.

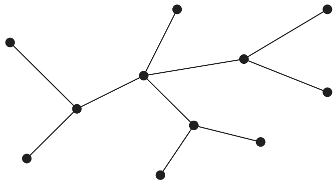

Figure 1.2 An example of a tree.

A tree $T = (V, E)$ is called *binary* if every interior vertex has degree 3, and it is called a *star tree* if it has precisely one interior vertex which then must be necessarily of degree $|V| - 1$. The unique interior vertex of a star tree will also be called the *central vertex* of that tree.

The tree in Figure 1.2 has one interior vertex of degree 4 and is, therefore, not binary. There are three cherries in this tree.

Note that, for any two distinct vertices $u$ and $v$ in a tree $T = (V, E)$, there is a unique edge $e^{v \to u} \in E(v)$ in the intersection $E_T(u|v) \cap E_v$. In consequence, there is precisely one path in $T$ from $u$ to $v$ for any two distinct vertices $u$ and $v$ of $T$ that we will denote by $\mathbf{p}_T(u, v)$ or simply by $\mathbf{p}(u, v)$, while its vertex set $V(\mathbf{p}_T(u, v))$ will be denoted by $V_T[u, v]$ and its edge set $\big($that actually coincides with $E_T(u|v)\big)$ also by $E_T[u, v]$. Clearly, a subset $U$ of $V$ is connected if and only if $V_T[u, v] \subseteq U$ holds for all $u, v \in U$ implying that, given any subset $U$ of $V$, there exists a unique smallest connected subset of $V$ that contains $U$, *viz.* the subset $V_T[U] := \bigcup_{u,v \in U} V_T[u, v]$. And we have $e^{v \to u} \neq e^{v \to u'}$ for three distinct vertices $u, u', v$ of $T$ if and only if $v \in V_T[u, u']$ holds.

Note that, for any three vertices $u, v, w$ in a tree $T = (V, E)$, there is a unique vertex $m \in V$ that is contained in the intersection $V_T[u, v] \cap V_T[v, w] \cap V_T[w, u]$, called the *median* of $u, v, w$ in $T$ and denoted by $med(u, v, w) = med_T(u, v, w)$.

Note also that, for every edge $e = \{u, v\}$ of a tree $T = (V, E)$, the subgraph $T^{(e)} = (V, E - e)$ of $T$ has precisely two connected components, *viz.* $T^{(e)}(u)$, the one containing $u$, and $T^{(e)}(v)$, the one containing $v$. Note also that $e \in E_T[u', v']$ holds for some edge $e \in E$ and any two vertices $u', v' \in V$ if and only if $T^{(e)}(u') \neq T^{(e)}(v')$ or, equivalently, $\pi_0(T^{(e)}) = \{T^{(e)}(u'), T^{(e)}(v')\}$ holds.

In particular, one has

$$E_T[u, w] = E_T[u, v] \triangle E_T[v, w] \subseteq E_T[u, v] \cup E_T[v, w] \tag{1.1}$$

for any three vertices $u, v, w$ of a tree $T = (V, E)$ (where $A \triangle B$ denotes, for any two sets $A, B$, their *symmetric difference* $A \cup B - A \cap B$) as $T^{(e)}(u) \neq T^{(e)}(v)$ holds for any edge $e \in E$ and any two vertices $u, v \in V$ if and only

$T^{(e)}(u) = T^{(e)}(w) \neq T^{(e)}(v)$ or $T^{(e)}(u) \neq T^{(e)}(w) = T^{(e)}(v)$ holds for any further vertex $w \in V$.

Further, a graph that, whether connected or not, at least contains no cycles is called a *forest*. Clearly, a graph $F = (V, E)$ is a forest if and only if the induced graph $F[U]$ is a tree for every connected component $U$ of $F$ and, hence, just as well for every connected subset $U$ of $V$. Note that a graph $G = (V, E)$ is a forest if and only if every edge $e = \{u, v\} \in E$ is a bridge.

In the context of graphs and trees, we will also follow popular practice and freely use the term *network* instead of the term *graph*, in particular when referring to connected graphs that are not (necessarily) trees.

A surjective map $\psi : V \to U$ from the vertex set $V$ of a graph $G = (V, E)$ onto another set $U$ is called a *contracting map* (for $G$) if all subsets of $V$ of the form $\psi^{-1}(u)$ $(u \in U)$ are connected. Clearly, given an equivalence relation $\sim$ on $V$, the canonical map $V \to V/\sim$ from $V$ onto the set $V/\sim$ of $\sim$-equivalence classes is a contracting map if and only if all $\sim$-equivalence classes are connected.

Further, given a graph $G = (V, E)$ and a contracting map $\psi : V \to U$ for $G$, we denote by $\psi G$ the graph with vertex set $U$ and edge set

$$\psi E := \{\{\psi(u), \psi(v)\} : \{u, v\} \in E, \psi(u) \neq \psi(v)\}.$$

Note that $\psi G$ is a tree whenever $G$ is a tree $T$, and that the map

$$\psi_\star : V \to \{\star\}\dot\cup(V - V_{int}(G), ) : v \mapsto \begin{cases} \star & \text{if } v \in V_{int}(G), \\ v & \text{otherwise,} \end{cases}$$

from $V$ onto the disjoint union of the set $V - V_{int}(G)$ of leaves of $G$ and just one additional element $\star$ not yet involved in $G$ is a contracting map if and only if $V_{int}(G)$ is a connected subset of $V$. So, this holds in particular whenever $G$ is connected in which case the resulting graph $\psi_\star G$ is a star tree.

We will say that a graph $G' = (V', E')$ results from a graph $G = (V, E)$ by the *contraction of an edge* $e \in E$ if $G' = \psi G$ holds for some contracting map $\psi : V \to V'$ that only contracts the edge $e$, i.e., for which all but one of the subsets of $V$ of the form $\psi^{-1}(v')$ $(v' \in V')$ have cardinality 1 while the unique remaining subset of $V$ of that form has cardinality 2 and is actually the edge $e$.

Clearly, such a contracting map exists for every edge $e \in E$. For example, the canonical map $\psi_e : V \to V/\sim_e$ from $V$ onto the set $V/\sim_e$ of equivalence classes $V/\sim_e$ of $V$ relative to the equivalence relation $\sim_e$ defined by

$$u \sim_e v \iff u = v \quad \text{or} \quad \{u, v\} = e$$

is a contracting map that just contracts $e$. And it is also obvious that every contracting map is a concatenation of such "elementary" contracting maps.

And if, even more specifically, $e$ is a pendant edge containing exactly one interior vertex $v$, we can choose this vertex as a canonical representative of the $\sim_e$-equivalence class $e \subseteq V$ of $v$ in $V/\sim_e$ and replace the map $\psi_e : V \to V/\sim_e$ by the map

$$\psi^e : V \to V - v : w \mapsto \begin{cases} v & \text{if } w \in e, \\ w & \text{else.} \end{cases}$$

Note that, denoting the unique leaf in $e$ by $u$, the graph $\psi^e G$ resulting from contracting the edge $e$ in this way coincides with the subgraph

$$G^e := (V - u, E - e)$$

of $G$ obtained by eliminating the pendant edge $e$ and the leaf $u$ in $e$.

Next, we state (without proof) some well-known simple facts that we will use later in this book.

**Lemma 1.2** (i) *Given any finite graph $G = (V, E)$, one has*

$$2|E| = |\{(v, e) \in V \times E : v \in e\}| = \sum_{i \geq 1} i\ |V^{(i)}|$$

*where $V^{(i)}$ denotes, for every $i \in \mathbb{N}_{\geq 0}$, the set*

$$V^{(i)} := \{v \in V : \deg(v) = i\}$$

*of vertices of degree $i$ in $G$.*

(ii) *A finite graph $G = (V, E)$ is a tree if and only if it is connected and $|V| = |E| + 1$ holds.*

(iii) *For every finite tree $T = (V, E)$ with at least two vertices, one has $\sum_{i \geq 1} (2 - i)|V^{(i)}| = 2$ or, equivalently,*

$$|V^{(1)}| = 2 + |V^{(3)}| + 2|V^{(4)}| + 3|V^{(5)}| + \cdots$$

*and, denoting by $V^{(i|j)}$ the set of vertices of degree $i$ that are adjacent to exactly $j$ leaves (which clearly is empty if $j < 0$ holds), one also has*

$$\sum_{i > 1} |V^{(i|i-1)}| = 2 + \sum_{i} |V^{(i|i-3)}| + 2 \sum_{i} |V^{(i|i-4)}| + 3 \sum_{i} |V^{(i|i-5)}| + \cdots$$

*provided $T$ is not a star tree: Indeed, if $T$ is not a star tree, the graph $\partial T := (V_{int}(T), E_{int}(T))$ derived from $T$ by deleting all of its leaves and pendant edges is a tree with at least two vertices, exactly $\sum_{i>1} |V^{(i|i-1)}|$ leaves, and, $\sum_i |V^{(i|i-j)}|$ vertices of degree $j$ in $\partial T$.*