Part I

Introduction to gravity and supergravity

Let no one ignorant of Mathematics enter here.

Inscription above the doorway of Plato's Academy

1
Differential geometry

The main purpose of this chapter is to fix our notation and to review the ideas and formulae of differential geometry we will make heavy use of. There are many excellent physicist-oriented references on differential geometry. Two that we particularly like are Refs. [481] and [972]. Our approach here will be quite pragmatic, ignoring many mathematical details and subtleties that can be found in the many excellent books on the subject.

1.1 World tensors

A *manifold* is a topological space that looks (i.e. it is homeomorphic to) locally (i.e. in a *patch*) like a piece of $\mathbb{R}^d$. d is the dimension of the manifold and the correspondence between the patch and the piece of $\mathbb{R}^n$ can be used to label the points in the patch by Cartesian $\mathbb{R}^n$ coordinates x^μ. In the overlap between different patches the different coordinates are consistently related by a *general coordinate transformation* (GCT) $x'^{\mu}(x)$. Only objects with good transformation properties under GCTs can be defined globally on the manifold. These objects are *tensors*.

A *contravariant vector field* (or $(1,0)$-type tensor or just "vector") $\xi(x) = \xi^\mu(x)\partial_\mu$ is defined at each point on a d-dimensional smooth manifold by its action on a function

$$\xi : f \longrightarrow \xi f = \xi^\mu \partial_\mu f, \tag{1.1}$$

which defines another function. These objects span a d-dimensional linear vector space at each point of the manifold called the *tangent space* $\mathrm{T}_p^{(1,0)}$. The d functions $\xi^\mu(x)$ are the vector components with respect to the *coordinate basis* $\{\partial_\mu\}$.

A *covariant vector field* (or $(0,1)$-type tensor or *differential 1-form*) is an element of the dual vector space (sometimes called the *cotangent space*) $\mathrm{T}_p^{(0,1)}$ and therefore transforms vectors into functions. The elements of the basis dual to the coordinate basis of contravariant vectors are usually denoted by $\{dx^\mu\}$ and, by definition,

$$\langle dx^\mu | \partial_\nu \rangle \equiv \delta^\mu{}_\nu, \tag{1.2}$$

which implies that the action of a form $\omega = \omega_\mu dx^\mu$ on a vector $\xi(x) = \xi^\mu(x)\partial_\mu$ gives the function[1]

$$\langle\omega|\xi\rangle = \omega_\mu \xi^\mu. \tag{1.3}$$

Under a GCT, vectors and forms transform as functions, i.e. $\xi'(x') = \xi(x(x'))$ etc., which means for their components in the associated coordinate basis

$$\frac{\partial x'^{\rho}}{\partial x^\mu}\,\xi^\mu(x(x')) = \xi'^{\rho}(x'), \qquad \omega_\mu(x(x'))\,\frac{\partial x^\mu}{\partial x'^{\rho}} = \omega'_{\rho}(x'). \tag{1.4}$$

More general tensors of type (q, r) can be defined as elements of the space $\mathrm{T}_p^{(q,r)}$, which is the tensor product of q copies of the tangent space and r copies of the cotangent space. Their components $\mathrm{T}^{\mu_1\cdots\mu_q}{}_{\nu_1\cdots\nu_r}$ transform under GCTs in the obvious way.

It is also possible to define *tensor densities of weight* w whose components in a coordinate basis change under a GCT with an extra factor of the Jacobian raised to the power $w/2$. Thus, for weight w, the vector density components $\mathfrak{v}^\mu$ and the form density components $\mathfrak{w}_\mu$ transform according to

$$\begin{aligned} \left|\frac{\partial x'}{\partial x}\right|^{w/2} \frac{\partial x'^{\rho}}{\partial x^\mu}\,\mathfrak{v}^\mu(x(x')) &= \mathfrak{v}'^{\rho}(x'), \\ \mathfrak{w}_\mu(x(x'))\,\frac{\partial x^\mu}{\partial x'^{\rho}}\left|\frac{\partial x'}{\partial x}\right|^{w/2} &= \mathfrak{w}'_{\rho}(x'), \end{aligned} \tag{1.5}$$

where for the Jacobian we use the notation

$$\left|\frac{\partial x'}{\partial x}\right| \equiv \det\left(\frac{\partial x'^{\rho}}{\partial x^\mu}\right). \tag{1.6}$$

An infinitesimal GCT[2] can be written as follows:

$$\delta x^\mu = x'^{\mu} - x^\mu = \epsilon^\mu(x). \tag{1.7}$$

The corresponding infinitesimal transformations of scalars ϕ and contravariant and covariant *world vectors* (an alternative name for components in the coordinate basis) are:[3]

$$\begin{aligned} \delta\phi &= -\epsilon^\lambda\partial_\lambda\phi &&\equiv -\mathcal{L}_\epsilon\phi, \\ \delta\xi^\mu &= -\epsilon^\lambda\partial_\lambda\xi^\mu + \partial_\nu\epsilon^\mu\xi^\nu &&\equiv -\mathcal{L}_\epsilon\xi^\mu \equiv -[\epsilon,\xi]^\mu, \\ \delta\omega_\mu &= -\epsilon^\lambda\partial_\lambda\omega_\mu - \partial_\mu\epsilon^\nu\omega_\nu &&\equiv -\mathcal{L}_\epsilon\omega_\mu, \end{aligned} \tag{1.8}$$

[1] Summation over repeated indices in any position will always be assumed, unless they are in parentheses.
[2] This is an element of a one-parameter group of GCTs (the unit element corresponding to the value 0 of the parameter) with a value of the parameter much smaller than 1.
[3] We use the functional variations $\delta\phi \equiv \phi'(x) - \phi(x)$ which refer to the value of the field ϕ at two different points whose coordinates are equal in the two different coordinate systems. They are denoted in Ref. [1068] by δ_0. They should be distinguished from the total variations $\tilde{\delta} = \phi'(x') - \phi(x)$ which refer to the values of the field ϕ at the same point in two different coordinate systems. The relation between the two is $\delta\phi = \tilde{\delta}\phi - \epsilon^\mu\partial_\mu\phi$. The piece $-\epsilon^\lambda\partial_\lambda\phi$ that appears in δ variations is the "transport term," which is not present in other kinds of infinitesimal variations. The transformations δ do enjoy a group property (their commutator is another δ transformation), whereas the transformations $\tilde{\delta}$ or the transport terms by themselves do not.

and, for weight-w scalar densities $\mathfrak{f}$, vector density components $\mathfrak{v}^\mu$, and the form density components $\mathfrak{w}_\mu$,

$$\begin{aligned} \delta\mathfrak{f} &= -\epsilon^\lambda\partial_\lambda\mathfrak{f} - w\partial_\lambda\epsilon^\lambda\mathfrak{f} &&\equiv -\mathcal{L}_\epsilon\mathfrak{f}, \\ \delta\mathfrak{v}^\mu &= -\epsilon^\lambda\partial_\lambda\mathfrak{v}^\mu + \partial_\nu\epsilon^\mu\mathfrak{v}^\nu - w\partial_\lambda\epsilon^\lambda\mathfrak{v}^\mu &&\equiv -\mathcal{L}_\epsilon\mathfrak{v}^\mu, \\ \delta\mathfrak{w}_\mu &= -\epsilon^\lambda\partial_\lambda\mathfrak{w}_\mu - \partial_\mu\epsilon^\nu\mathfrak{w}_\nu - w\partial_\lambda\epsilon^\lambda\mathfrak{w}_\mu &&\equiv -\mathcal{L}_\epsilon\mathfrak{w}_\mu, \end{aligned} \tag{1.9}$$

where $\mathcal{L}_\epsilon$ is the *Lie derivative* with respect to the vector field ϵ and $[\epsilon,\xi]$ is the *Lie bracket* of the vectors ϵ and ξ. The definition of the Lie derivative can be extended to tensors or weight-w tensor densities of any type:

$$\begin{aligned} \mathcal{L}_\epsilon T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} &= -\delta_\epsilon T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} \\ &= \epsilon^\rho\partial_\rho T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} - \partial_\rho\epsilon^{\mu_1}T^{\rho\mu_2\cdots\mu_p}{}_{\nu_1\cdots\nu_q} + \cdots \\ &\quad + \partial_{\nu_1}\epsilon^\rho T^{\mu_1\cdots\mu_p}{}_{\rho\nu_2\cdots\nu_q} - w\partial_\lambda\epsilon^\lambda T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}. \end{aligned} \tag{1.10}$$

In particular the metric (a symmetric $(0,2)$-type tensor to be defined later) and r-forms (a fully antisymmetric $(0,r)$-type tensor) transform as follows:

$$\begin{aligned} \delta g_{\mu\nu} &= -\epsilon^\lambda\partial_\lambda g_{\mu\nu} - 2g_{\lambda(\mu}\partial_{\nu)}\epsilon^\lambda &&= -\mathcal{L}_\epsilon g_{\mu\nu}, \\ \delta B_{\mu_1\cdots\mu_r} &= -\epsilon^\lambda\partial_\lambda B_{\mu_1\cdots\mu_r} - r(\partial_{[\mu_1|}\epsilon^\lambda)B_{\lambda|\mu_2\cdots\mu_r]} &&= -\mathcal{L}_\epsilon B_{\mu_1\cdots\mu_r}. \end{aligned} \tag{1.11}$$

The main properties of the Lie derivative are that it transforms tensors of a given type into tensors of the same given type, it obeys the Leibniz rule $\mathcal{L}_\epsilon(T_1T_2) = (\mathcal{L}_\epsilon T_1)T_2 + T_1\mathcal{L}_\epsilon T_2$, it is connection independent, and it is linear with respect to ϵ. Furthermore, it satisfies the *Jacobi identity*

$$[\mathcal{L}_{\xi_1},[\mathcal{L}_{\xi_2},\mathcal{L}_{\xi_3}]] + [\mathcal{L}_{\xi_2},[\mathcal{L}_{\xi_3},\mathcal{L}_{\xi_1}]] + [\mathcal{L}_{\xi_3},[\mathcal{L}_{\xi_1},\mathcal{L}_{\xi_2}]] = 0, \tag{1.12}$$

where the brackets stand for commutators of differential operators. The relation between the commutator $[\mathcal{L}_\xi,\mathcal{L}_\epsilon]$ and the Lie bracket $[\xi,\epsilon]$ is

$$[\mathcal{L}_\xi,\mathcal{L}_\epsilon] = \mathcal{L}_{[\xi,\epsilon]}. \tag{1.13}$$

Thus, the Lie bracket is an antisymmetric, bilinear product in tangent space that also satisfies the Jacobi identity

$$[\xi_1,[\xi_2,\xi_3]] + [\xi_2,[\xi_3,\xi_1]] + [\xi_3,[\xi_1,\xi_2]] = 0, \tag{1.14}$$

which one can use to give it the structure of *Lie algebra*.

1.2 Affinely connected spacetimes

The *covariant derivative* of world tensors is defined by

$$\begin{aligned} \nabla_\mu\phi &= \partial_\mu\phi, \\ \nabla_\mu\xi^\nu &= \partial_\mu\xi^\nu + \Gamma_{\mu\rho}{}^\nu\xi^\rho, \\ \nabla_\mu\omega_\nu &= \partial_\mu\omega_\nu - \omega_\rho\Gamma_{\mu\nu}{}^\rho, \end{aligned} \tag{1.15}$$

and on weight-w tensor densities by

$$\begin{aligned}\nabla_\mu \mathfrak{f} &= \partial_\mu \mathfrak{f} - w\Gamma_{\mu\rho}{}^{\rho}\mathfrak{f},\\ \nabla_\mu \mathfrak{v}^\nu &= \partial_\mu \mathfrak{v}^\nu + \Gamma_{\mu\rho}{}^{\nu}\mathfrak{v}^\rho - w\Gamma_{\mu\rho}{}^{\rho}\mathfrak{v}^\nu,\\ \nabla_\mu \mathfrak{w}_\nu &= \partial_\mu \mathfrak{w}_\nu - \mathfrak{w}_\rho\Gamma_{\mu\nu}{}^{\rho} - w\Gamma_{\mu\rho}{}^{\rho}\mathfrak{w}_\nu,\end{aligned} \tag{1.16}$$

where Γ is the *affine connection*, and is added to the partial derivative so that the covariant derivative of a tensor transforms as a tensor in all indices. This requires the affine connection to transform under infinitesimal GCTs as follows:

$$\delta\Gamma_{\mu\nu}{}^{\rho} = -\mathcal{L}_\epsilon\Gamma_{\mu\nu}{}^{\rho} - \partial_\mu\partial_\nu\epsilon^\rho, \tag{1.17}$$

and therefore it is not a tensor. In principle it can be any field with the above transformation properties and should be understood as structure added to our manifold. A d-dimensional manifold equipped with an affine connection is sometimes called an *affinely connected space* and is denoted by L_d.

The definition of a covariant derivative can be extended to tensors of arbitrary type in the standard fashion. Its main properties are that it is a linear differential operator that transforms type-(p, q) tensors into $(p, q + 1)$ tensors (hence the name covariant) and obeys the Leibniz rule and the Jacobi identity.

Let us now decompose the connection into two (symmetric and antisymmetric) pieces under the exchange of the covariant indices:

$$\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{(\mu\nu)}{}^{\rho} + \Gamma_{[\mu\nu]}{}^{\rho}. \tag{1.18}$$

The antisymmetric part is called the *torsion* and it is a tensor (which the connection is not)

$$T_{\mu\nu}{}^{\rho} = -2\Gamma_{[\mu\nu]}{}^{\rho}. \tag{1.19}$$

As we have said, the Lie derivative transforms tensors into tensors in spite of the fact that it is expressed in terms of partial derivatives. We can rewrite it in terms of covariant derivatives and torsion terms to make evident the fact that the result is indeed a tensor:

$$\begin{aligned}\mathcal{L}_\epsilon\phi &= \epsilon^\lambda\nabla_\lambda\phi,\\ \mathcal{L}_\epsilon\xi^\mu &= \epsilon^\lambda\nabla_\lambda\xi^\mu - \nabla_\nu\epsilon^\mu\xi^\nu + \epsilon^\lambda T_{\lambda\rho}{}^{\mu}\xi^\rho,\\ \mathcal{L}_\epsilon\omega_\mu &= \epsilon^\lambda\nabla_\lambda\omega_\mu + \nabla_\mu\epsilon^\nu\omega_\nu - \epsilon^\lambda\omega_\rho T_{\lambda\mu}{}^{\rho},\end{aligned} \tag{1.20}$$

etc. It should be stressed that this is just a rewriting of the Lie derivative, which is independent of any connection. There are other connection-independent derivatives. Particularly important is the *exterior derivative* defined on *differential forms* (completely antisymmetric tensors) which we will study later in Section 1.7.

The additional structure of an affine connection allows us to define *parallel transport*. In a generic spacetime there is no natural notion of parallelism for two vectors defined at two different points. We need to transport one of them keeping it "parallel to itself" to the point at which the other is defined. Then we can compare the two vectors at the same point. Using the affine connection, we can define an infinitesimal *parallel displacement* of a covariant vector ω_μ in the direction of ϵ^μ by

$$\delta_{\mathrm{P}_\epsilon}\omega_\mu = \epsilon^\nu\Gamma_{\nu\mu}{}^{\rho}\omega_\rho. \tag{1.21}$$

If $\omega_\mu(x)$ is a vector field, we can compare its value at a given point $x^\mu + \epsilon^\mu$ with the value obtained by parallel displacement from x^μ. The difference is precisely given by the covariant derivative in the direction ϵ^μ:

$$\omega_\mu(x') - (\omega_\mu + \delta_{\mathrm{P}_\epsilon}\omega_\mu)(x) = \epsilon^\nu \nabla_\nu \omega_\mu. \tag{1.22}$$

A vector field whose value at every point coincides with the value one would obtain by parallel transport from neighboring points is a *covariantly constant* vector field, $\nabla_\nu \omega_\mu = 0$.

If the vector tangential to a curve[4] $v^\mu = dx^\mu/d\xi \equiv \dot{x}^\mu$ is parallel to itself along the curve (as a straight line in flat spacetime) then

$$v^\nu \nabla_\nu v^\mu = \ddot{x}^\mu + \dot{x}^\rho \dot{x}^\sigma \Gamma_{\rho\sigma}{}^\mu = 0, \tag{1.23}$$

which is the *autoparallel equation*. This is the equation satisfied by an *autoparallel curve*, which is the generalization of a straight line to a general affinely connected spacetime. There is a second possible generalization based on the property of straight lines of being the shortest possible curves joining two given points (*geodesics*), but it requires the notion of length and we will have to wait until the introduction of metrics.

We can understand the meaning of torsion using parallel transport: let us consider two vectors ϵ_1^μ and ϵ_2^μ at a given point of coordinates x^μ. Let us now consider at the point of coordinates $x^\mu + \epsilon_1^\mu$ the vector $\epsilon_2'^\mu$ obtained by parallel-transporting ϵ_2^μ in the direction ϵ_1^μ and, at the point of coordinates $x^\mu + \epsilon_2^\mu$, the vector $\epsilon_1'^\mu$ obtained by parallel-transporting ϵ_1^μ in the direction ϵ_2^μ. In flat spacetime, the vectors $\epsilon_1, \epsilon_2, \epsilon_1'$, and ϵ_2' form an infinitesimal parallelogram since $x^\mu + \epsilon_1^\mu + \epsilon_2'^\mu = x^\mu + \epsilon_2^\mu + \epsilon_1'^\mu$. In a general affinely connected spacetime, the infinitesimal parallelogram does not close and

$$\left(x^\mu + \epsilon_1^\mu + \epsilon_2'^\mu\right) - \left(x^\mu + \epsilon_2^\mu + \epsilon_1'^\mu\right) = \epsilon_1^\rho \epsilon_2^\sigma T_{\rho\sigma}{}^\mu. \tag{1.24}$$

Finite parallel transport along a curve γ depends on the curve, not only on the initial and final points, so, if the curve is closed, the original and the parallel-transported vectors do not coincide. The difference is measured by the *(Riemann) curvature tensor* $R_{\mu\nu\rho}{}^\sigma$: let us consider two vectors ϵ_1^μ and ϵ_2^μ at a given point x^μ and let us parallel-transport the vector ω_μ from x^μ to $x^\mu + \epsilon_1^\mu$ and then to $x^\mu + \epsilon_1^\mu + \epsilon_2^\mu$. The result is

$$\omega_\mu + (\epsilon_1^\nu + \epsilon_2^\nu)\Gamma_{\nu\mu}{}^\rho \omega_\rho + \epsilon_1^\lambda \epsilon_2^\nu \left(\partial_\lambda \Gamma_{\nu\mu}{}^\rho + \Gamma_{\lambda\delta}{}^\rho \Gamma_{\nu\mu}{}^\delta\right)\omega_\rho + \mathcal{O}(\epsilon^3). \tag{1.25}$$

If we go to the same point along the route x^μ to $x^\mu + \epsilon_2^\mu$ and then to $x^\mu + \epsilon_1^\mu + \epsilon_2^\mu$ we obtain a different value, and the difference between the parallel-transported vectors is

$$\Delta\omega_\mu = \epsilon_1^\lambda \epsilon_2^\nu R_{\lambda\nu\mu}{}^\rho \omega_\rho, \tag{1.26}$$

where

$$R_{\mu\nu\rho}{}^\sigma(\Gamma) = 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^\sigma + 2\Gamma_{[\mu|\lambda}{}^\sigma \Gamma_{|\nu]\rho}{}^\lambda. \tag{1.27}$$

[4] Here we use the mathematical concept of a curve: a map from the real line $\mathbb{R}$ (or an interval) given as a function of a real parameter $x^\mu(\xi)$, rather than the *image* of the real line in the spacetime. Thus, after a reparametrization $\xi'(\xi)$, we obtain a different curve, although the image is the same and physically we would say that we have the same curve.

We can also define the curvature tensor (and the torsion tensor) through the *Ricci identities* for a scalar ϕ, a vector ξ^μ, and a 1-form ω_μ:

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]\,\phi &= T_{\mu\nu}{}^\sigma \nabla_\sigma \phi, \\
[\nabla_\mu, \nabla_\nu]\,\xi^\rho &= R_{\mu\nu\sigma}{}^\rho \xi^\sigma + T_{\mu\nu}{}^\sigma \nabla_\sigma \xi^\rho, \\
[\nabla_\mu, \nabla_\nu]\,\omega_\rho &= -\omega_\sigma R_{\mu\nu\rho}{}^\sigma + T_{\mu\nu}{}^\sigma \nabla_\sigma \omega_\rho,
\end{aligned} \tag{1.28}$$

or, for a general tensor,

$$[\nabla_\alpha, \nabla_\beta]\,\xi_{\mu_1\cdots}{}^{\nu_1\cdots} = -R_{\alpha\beta\mu_1}{}^\gamma \xi_{\gamma\cdots}{}^{\nu_1\cdots} - \cdots + R_{\alpha\beta\gamma}{}^{\nu_1} \xi_{\mu_1\cdots}{}^{\gamma\cdots} + \cdots + T_{\alpha\beta}{}^\gamma \nabla_\gamma \xi_{\mu_1\cdots}{}^{\nu_1\cdots}, \tag{1.29}$$

and, using the antisymmetry of the commutators of covariant derivatives and the fact that the covariant derivative satisfies the Jacobi identity, one can derive the following *Bianchi identities*:

$$\begin{aligned}
R_{(\alpha\beta)\gamma}{}^\delta &= 0, \\
R_{[\alpha\beta\gamma]}{}^\delta + \nabla_{[\alpha} T_{\beta\gamma]}{}^\delta + T_{[\alpha\beta}{}^\rho T_{\gamma]\rho}{}^\delta &= 0, \\
\nabla_{[\alpha} R_{\beta\gamma]\rho}{}^\sigma + T_{[\alpha\beta}{}^\delta R_{\gamma]\delta\rho}{}^\sigma &= 0.
\end{aligned} \tag{1.30}$$

(The last two identities are derived from the Jacobi identity of covariant derivatives acting on a scalar and a vector, respectively.)

In general, if we modify the affine connection by adding an arbitrary tensor[5] $\tau_{\mu\nu}{}^\rho$,

$$\Gamma_{\mu\nu}{}^\rho \to \tilde{\Gamma}_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho + \tau_{\mu\nu}{}^\rho, \tag{1.31}$$

the curvature is modified as follows:

$$R_{\mu\nu\rho}{}^\sigma(\tilde{\Gamma}) = R_{\mu\nu\rho}{}^\sigma(\Gamma) - T_{\mu\nu}{}^\lambda \tau_{\lambda\rho}{}^\sigma + 2\nabla_{[\mu} \tau_{\nu]\rho}{}^\sigma + 2\tau_{[\mu|\lambda}{}^\sigma \tau_{|\nu]\rho}{}^\lambda. \tag{1.32}$$

The *Ricci tensor* is defined by

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho = \partial_\mu \Gamma_{\rho\nu}{}^\rho - \partial_\rho \Gamma_{\mu\nu}{}^\rho + \Gamma_{\mu\lambda}{}^\rho \Gamma_{\rho\nu}{}^\lambda - \Gamma_{\rho\lambda}{}^\rho \Gamma_{\mu\nu}{}^\lambda. \tag{1.33}$$

In general it is not symmetric, but, according to the second Bianchi identity,

$$R_{[\mu\nu]} = \tfrac{1}{2} \overset{*}{\nabla}_\rho \overset{*}{T}_{\mu\nu}{}^\rho + \tfrac{1}{2} R_{\mu\nu\rho}{}^\rho, \tag{1.34}$$

where we have used the *modified divergence* $\overset{*}{\nabla}_\mu$ and the *modified torsion tensor* $\overset{*}{T}_{\mu\nu}{}^\rho$,

$$\overset{*}{\nabla}_\mu = \nabla_\mu - T_{\mu\rho}{}^\rho, \qquad \overset{*}{T}_{\mu\nu}{}^\rho = T_{\mu\nu}{}^\rho - 2T_{[\mu|\sigma}{}^\sigma \delta_{|\nu]}{}^\rho. \tag{1.35}$$

If we modify the connection as in Eq. (1.31), the Ricci tensor is also modified:

$$R_{\mu\rho}(\tilde{\Gamma}) = R_{\mu\rho} - T_{\mu\nu}{}^\lambda \tau_{\lambda\rho}{}^\nu + 2\nabla_{[\mu} \tau_{\nu]\rho}{}^\nu + 2\tau_{[\mu|\lambda}{}^\nu \tau_{|\nu]\rho}{}^\lambda. \tag{1.36}$$

Another useful formula is the Lie derivative of the torsion tensor which, using the first two Bianchi identities, can be rewritten in the form

$$\mathcal{L}_\xi T_{\mu\nu}{}^\rho = \nabla_\mu \left(\xi^\lambda T_{\lambda\nu}{}^\rho\right) + \nabla_\nu \left(\xi^\lambda T_{\mu\lambda}{}^\rho\right) - \nabla_\lambda \left(\xi^\rho T_{\mu\nu}{}^\lambda\right) - 3\xi^\lambda R_{[\lambda\mu\nu]}{}^\rho + \xi^\sigma \nabla_\sigma T_{\mu\nu}{}^\rho. \tag{1.37}$$

[5] Only if τ transforms as a tensor can $\tilde{\Gamma}$ transform as a connection.

1.3 Metric spaces

To go further we need to add structure to a manifold: a *metric* in tangent space, i.e. an inner product for tangent-space vectors (symmetric, bilinear) associating a function $g(\xi, \epsilon)$ with any pair of vectors (ξ, ϵ). This corresponds to a symmetric $(0, 2)$-type tensor g symmetric in its two covariant components $g_{\mu\nu} = g_{(\mu\nu)}$:

$$\xi \cdot \epsilon \equiv g(\xi, \epsilon) = \xi^{\mu} \epsilon^{\nu} g_{\mu\nu}. \tag{1.38}$$

The norm squared of a vector is just the product of the vector with itself, $\xi^2 = \xi \cdot \xi$. The metric will be required to be non-singular, i.e.

$$g \equiv \det(g_{\mu\nu}) \neq 0, \tag{1.39}$$

and locally diagonalizable into $\eta_{\mu\nu} = \mathrm{diag}(+ - \cdots -)$ for physical and conventional reasons. Thus, in d dimensions

$$\mathrm{sign}\ g = \frac{g}{|g|} = (-1)^{d-1}. \tag{1.40}$$

As usual, a metric can be used to establish a correspondence between a vector space and its dual, i.e. between vectors and 1-forms: with each vector ξ^{μ} we associate a 1-form ω_{μ} whose action on any other vector η^{μ} is the product of ξ and η, $\omega(\eta) = \xi^{\mu}\eta^{\nu} g_{\mu\nu}$, which means the relation between components $\omega_{\nu} = \xi^{\mu} g_{\mu\nu}$. It is customary to denote this 1-form by ξ_{μ}, and the transformation from vector to 1-form is represented by lowering the index.

The inverse metric can be used as a metric in cotangent space, and its components are those of the inverse matrix and are denoted with upper indices. The operation of raising indices can be similarly defined, and the consistency of all these operations is guaranteed because the dual of the dual is the original vector space. The extension to tensors of higher ranks is straightforward.

The determinant of the metric can also be used to relate tensors and weight w tensor densities, since it transforms as a density of weight $w = 2$ and the product of a tensor and $g^{w/2}$ transforms as a density of weight w.

Furthermore, with a metric we can define the *Ricci scalar* R and the *Einstein tensor* $G_{\mu\nu}$,

$$R = R_{\mu}{}^{\mu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R, \tag{1.41}$$

which needs not be symmetric (just like the Ricci tensor).

So far we have two independent fields defined on our manifold: the metric and the affine connection. An L_d spacetime equipped with a metric is sometimes denoted by (L_d, g). The affine connection and the metric are related by the *non-metricity tensor* $Q_{\mu\nu\rho}$,

$$Q_{\mu\nu\rho} \equiv -\nabla_{\mu} g_{\nu\rho}. \tag{1.42}$$

If we take the combination $\nabla_{\mu} g_{\rho\sigma} + \nabla_{\rho} g_{\sigma\mu} - \nabla_{\sigma} g_{\mu\rho}$ and expand it, we find that the connection can be written as follows:

$$\Gamma_{\mu\nu}{}^{\rho} = \left\{ {\rho \atop \mu\,\nu} \right\} + K_{\mu\nu}{}^{\rho} + L_{\mu\nu}{}^{\rho}, \tag{1.43}$$

where

$$\left\{ {\rho \atop \mu\,\nu} \right\} = \tfrac{1}{2} g^{\rho\sigma}\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\} \tag{1.44}$$

are the *Christoffel symbols*, which are completely determined by the metric, and K is called the *contorsion tensor* and is given in terms of the torsion tensor by

$$\begin{aligned} K_{\mu\nu}{}^{\rho} &= \tfrac{1}{2} g^{\rho\sigma}\{T_{\mu\sigma\nu} + T_{\nu\sigma\mu} - T_{\mu\nu\sigma}\}\,, \\ K_{[\mu\nu]}{}^{\rho} &= -\tfrac{1}{2} T_{\mu\nu}{}^{\rho}, \qquad K_{\mu\nu\rho} = -K_{\mu\rho\nu}. \end{aligned} \tag{1.45}$$

Finally

$$L_{\mu\nu}{}^{\rho} = \tfrac{1}{2}\{Q_{\mu\nu}{}^{\rho} + Q_{\nu\mu}{}^{\rho} - Q^{\rho}{}_{\mu\nu}\}\,. \tag{1.46}$$

Observe that the contorsion tensor depends on the metric whereas the torsion tensor does not. Furthermore, observe that, since the contorsion and non-metricity tensors transform as tensors, the piece responsible for the non-homogeneous term in the transformation of the affine connection is the Christoffel symbol.

With a metric it is also possible to define the length of a curve γ, $x^\mu(\xi)$, by the integral

$$s = \int_\gamma d\xi\, \sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}. \tag{1.47}$$

If we consider the above expression as a functional in the space of all curves joining two given points, we can ask which of those curves minimizes it. The answer is given by the Euler–Lagrange equations, which take the simple form

$$\ddot{x}^\mu + \dot{x}^\rho \dot{x}^\sigma \left\{ {\mu \atop \rho\,\sigma} \right\} = 0, \tag{1.48}$$

if we parametrize the curve by its proper length s. This is the *geodesic equation*, and is different from the autoparallel equation (1.23) whenever there is torsion and non-metricity.

In the standard theory of gravity metric and affine connection are not independent variables since we want to describe only the degrees of freedom corresponding to a massless spin-2 particle. To relate these two fields one imposes the *metric postulate*

$$Q_{\mu\rho\sigma} = -\nabla_\mu g_{\rho\sigma} = 0, \tag{1.49}$$

which makes the operations of raising and lowering of indices commute with the covariant derivative. A connection satisfying the above condition is said to be *metric compatible* and a spacetime (L_d, g) with a metric-compatible connection is called a *Riemann–Cartan spacetime* and is denoted by U_d.

Sometimes a weaker condition is required: the vanishing of the trace-free part of the non-metricity tensor $\hat{Q}$

$$\hat{Q}_{\mu\nu\rho} \equiv Q_{\mu\nu\rho} - \tfrac{1}{d} Q_{\mu\sigma}{}^{\sigma} g_{\nu\rho} = 0. \tag{1.50}$$

In this case, the non-metricity must take the form

$$Q_{\mu\nu\rho} = -A_\mu g_{\nu\rho}, \quad \Rightarrow \quad L^{\rho}_{\mu\nu} = A_{(\mu} g_{\nu)}{}^{\rho} - \tfrac{1}{2} g_{\mu\nu} A^\rho, \tag{1.51}$$