1

Representations and modules

1.1 Basic concepts

In this section we introduce the basic concepts of representation theory, fix most of the notation used in this book and give a large number of concrete examples for representations.

Definition 1.1.1 A **representation** of a group G over a field K is a homomorphism $\delta\colon G \to \mathrm{GL}(V)$ of G into the group of invertible K-endomorphisms of a finite dimensional vector space V over K. The dimension of this vector space is called the **degree** of δ. A **matrix representation** of G of degree n is a homomorphism $\boldsymbol{\delta}\colon G \to \mathrm{GL}_n(K)$ of G into the full linear group $\mathrm{GL}_n(K)$ over K of some degree n. If δ or $\boldsymbol{\delta}$ is injective, it is called **faithful**.

If V is a K-vector space of dimension n and $B := (v_1, \ldots, v_n)$ is a K-basis of V, then by assigning to each endomorphism the matrix representing it with respect to the basis B one obtains a group isomorphism

$$\mathrm{GL}(V) \to \mathrm{GL}_n(K), \quad \varphi \mapsto [\varphi]_B,$$

where the matrix $[\varphi]_B = [a_{ij}] \in K^{n\times n}$ is defined by

$$\varphi(v_j) = \sum_{i=1}^{n} a_{ij} v_i \qquad (1 \le j \le n).$$

Thus for any representation $\delta\colon G \to \mathrm{GL}(V)$ and any choice of a basis B of V one obtains a matrix representation

$$\boldsymbol{\delta}_B\colon G \to \mathrm{GL}_n(K), \quad g \mapsto [\delta(g)]_B.$$

Observe that the multiplication in $\mathrm{GL}(V)$ and also in the ring $\mathrm{End}_K V$ of all K-endomorphisms of V is defined by $\varphi \circ \psi(v) = \varphi(\psi(v))$ for $v \in V$. We will usually omit the symbol "$\circ$."

Definition 1.1.2 Two representations $\delta\colon G \to \mathrm{GL}(V)$ and $\delta'\colon G \to \mathrm{GL}(W)$ are called **equivalent** if there is a K-vector space isomorphism $\varphi\colon V \to W$ such that

$$\delta'(g) = \varphi\,\delta(g)\,\varphi^{-1} \quad \text{for all} \quad g \in G.$$

Similarly two matrix representations $\boldsymbol{\delta}\colon G \to \mathrm{GL}_n(K)$ and $\boldsymbol{\delta}'\colon G \to \mathrm{GL}_n(K)$ are called **equivalent** if there is a matrix $T \in \mathrm{GL}_n(K)$ such that

$$\boldsymbol{\delta}'(g) = T\,\boldsymbol{\delta}(g)\,T^{-1} \quad \text{for all} \quad g \in G.$$

So obviously different matrix representations corresponding to the same representation are equivalent.

It is convenient to use the language of modules over rings or better yet of modules over algebras over a commutative ring K. In this book "ring" means *associative ring* having a unit element, which we denote by 1, or 1_A if the ring is called A. For convenience we recall some of the relevant definitions.

Definition 1.1.3 If K is a commutative ring, a **K-algebra** is a ring A together with a ring homomorphism $\lambda_A\colon K \to \mathbf{Z}(A)$ satisfying $\lambda_A(1_K) = 1_A$. Here $\mathbf{Z}(A)$ is the **center** of A, defined by

$$\mathbf{Z}(A) := \{z \in A \mid a\,z = z\,a \ \text{ for all } \ a \in A\}.$$

If (A, λ_A), $(A', \lambda_{A'})$ are K-algebras, then a ring homomorphism $\varphi\colon A \to A'$ is a K-algebra homomorphism if $\lambda_{A'} = \varphi \circ \lambda_A$.

Observe that any ring can be considered as a $\mathbb{Z}$-algebra in a unique way.

Definition 1.1.4 If A is a ring, an **A-module**, or, more precisely, a left A-module, is an abelian group V together with a map $A \times V \to V$, $(a, v) \mapsto a \cdot v$, satisfying

$$(a+b)\cdot v = a\cdot v + b\cdot v, \quad a\cdot(v+w) = a\cdot v + a\cdot w, \quad 1_A\cdot v = v \quad \text{for } a, b \in A,\ v, w \in V.$$

A right A-module is defined similarly with a map $V \times A \to V$, $(v, a) \mapsto v \cdot a$. It is equivalent to an A^{op}-module, where A^{op} stands for the **opposite ring** of A with multiplication changed to $a * a' := a' \cdot a$.

Remark 1.1.5 A together with the ring multiplication $A \times A \to A$ is an A-module, called the **left regular A-module**, often written as ${}_AA$. If (A, λ_A) is a K-algebra over a commutative ring K, then any A-module V becomes also a K-module by defining $\alpha \cdot v := \lambda_A(\alpha) \cdot v$ for $\alpha \in K$ and $v \in V$ with

$$\alpha \cdot (a \cdot v) = a \cdot (\alpha \cdot v) = (\lambda_A(\alpha)\,a) \cdot v \qquad \text{for} \qquad \alpha \in K,\ a \in A,\ v \in V\,.$$

In particular, for $V = {}_AA$ and $a = v = 1_A$ we get $\lambda_A(\alpha) = \alpha \cdot 1_A$. This is the reason why the notation λ_A will hardly ever be used and we will usually talk about an algebra A over K instead of (A, λ_A), regarding A as a K-module with $\alpha \cdot 1_A \in \mathbf{Z}(A)$ for all $\alpha \in K$. Occasionally we take the liberty to abbreviate $av := a \cdot v$ and $\alpha v := \alpha \cdot v$.

Important examples of algebras are group algebras. For simplicity we restrict ourselves to finite groups.

Definition 1.1.6 Let K be a commutative ring and $(G,\cdot)$ be a finite group. Put $KG := K^G$, the set of all maps from G to K, and define for $a,b \in KG$, $\alpha \in K$ and $g \in G$

$$(a+b)(g) := a(g)+b(g), \quad (ab)(g) := \sum_{h\in G} a(h)b(h^{-1}\cdot g), \quad (\alpha a)(g) := \alpha a(g).$$

For $g \in G$ let $g^o \in KG$ be defined by

$$g^o(h) := \begin{cases} 0 & \text{for } h \in G \setminus \{g\}, \\ 1_K & \text{for } h = g \end{cases}$$

Then it is readily verified that KG is a K-algebra with unit 1^o, where 1 is the unit in G. It is called the **group algebra** of G over K.

Remark 1.1.7 If $a \in KG$ then clearly $a = \sum_{g\in G} a(g)g^o$. Thus KG is a free K-module with basis $(g^o)_{g\in G}$, that is, every element $a \in KG$ can be uniquely written in the form $a = \sum_{g\in G} \alpha_g\, g^o$ with $\alpha_g \in K$. Also $g^o h^o = (gh)^o$, so that $g \mapsto g^o$ gives an embedding (= injective group homomorphism) from G to the group of units $(KG)^\times$ (= multiplicative group of invertible elements) of KG. It is common practice to identify $g \in G$ with $g^o \in KG$. Then

$$KG = \{\sum_{g\in G} \alpha_g\, g \mid \alpha_g \in K\}.$$

Of course, KG is commutative if and only if G is abelian.

Example 1.1.8 We consider the symmetric group $\mathrm{S}_n = (\mathrm{S}_n, \circ)$ of degree n, the set of all permutations of $\{1,\dots,n\}$ with multiplication $\circ$ defined by $(\sigma\circ\tau)(i) := \sigma(\tau(i))$ for $\sigma,\tau \in \mathrm{S}_n$ and $i \in \{1,\dots,n\}$. We will use the familiar cycle notation for elements of S_n. So, for instance, $\sigma = (2,3)(5,7,8) \in \mathrm{S}_9$ is the permutation with $\sigma(2) = 3$, $\sigma(3) = 2$, $\sigma(5) = 7$, $\sigma(7) = 8$, $\sigma(8) = 5$ and $\sigma(i) = i$ for $i \in \{1,4,6,9\}$. If K is a commutative ring then

$$K\,\mathrm{S}_3 = \{\alpha_1() + \alpha_2(1,2,3) + \alpha_3(1,3,2) + \alpha_4(1,2) + \alpha_5(2,3) + \alpha_6(1,3) \mid \alpha_i \in K\},$$

and in $K\,\mathrm{S}_3$ we can compute, for instance with $a := (\,() + (1,2,3) + (1,3,2)\,)$,

$$a\cdot(1,2) = (1,2) + (1,3) + (2,3), \qquad a\cdot(\,() - (1,3,2)\,) = 0, \qquad a^2 = 3\,a.$$

Assumption: For the rest of this section we will assume that G is a finite group and K a commutative ring. If we require that K is a field we will say so, unless it is clear from the context, e.g. if we are talking about K-vector spaces.

Notation: If M is a subset of G we put

$$M^+ := \sum_{g\in M} g \in KG. \tag{1.1}$$

As usual we write $g^h := h^{-1}gh$ for $g,h \in G$. Furthermore $g^G := \{g^h \mid h \in G\}$ and $\mathrm{cl}(G) := \{g^G \mid g \in G\}$, the set of conjugacy classes of G.

The center of a group algebra is easily described as follows.

Lemma 1.1.9 *$(C^+)_{C\in\mathrm{cl}(G)}$ is a K-basis of the center $\mathbf{Z}(KG)$ of KG.*

Proof. If $z=\sum_{g\in G}\alpha_g g\in KG$, then $z\in\mathbf{Z}(KG)$ if and only if $h^{-1}zh=z$ for all $h\in G$, so if and only if

$$\sum_{g\in G}\alpha_g h^{-1}gh=\sum_{g\in G}\alpha_{hgh^{-1}}g=\sum_{g\in G}\alpha_g g;$$

hence if and only if $\alpha_g=\alpha_{hgh^{-1}}$ for all $h,g\in G$. This condition is equivalent to saying that α_g must be constant on conjugacy classes, or in other words that $z=\sum_{C\in\mathrm{cl}(G)}\alpha_C C^+$, where $\alpha_C=\alpha_g$ if $g\in C$. □

Any representation $\delta\colon G\to\mathrm{GL}(V)$ of a finite group G over a field K (and likewise any matrix representation $\boldsymbol{\delta}\colon G\to\mathrm{GL}_n(K)$) extends by K-linearity naturally to a K-algebra homomorphism $\delta\colon KG\to\mathrm{End}_K V$ (resp. $\boldsymbol{\delta}\colon KG\to K^{n\times n}$), which will be denoted by the same symbol and which is called a representation (resp. matrix representation) of the group algebra. Also V becomes a (left) KG-module via $a\cdot v:=\delta(a)(v)$ for $a\in KG$ and $v\in V$. Conversely, if V is any KG-module, which has finite dimension as a K-vector space, then one obtains a representation $\delta\colon KG\to\mathrm{End}_K V$ by defining $\delta(a)(v):=a\cdot v$ and one obtains a representation of G by restricting δ to G. The KG-module V is often called the **representation module** of the representation $\delta\colon G\to\mathrm{GL}(V)$ and δ is called the representation "afforded by V." Obviously, equivalent representations have representation modules that are isomorphic as KG-modules and vice versa.

Although we are usually considering representations (and algebras) over a field, there are occasions where representations (and algebras) over a commutative ring K come up. In this case we will make provisions that the K-modules considered have a finite K-basis.

Definition 1.1.10 Let K be a commutative ring and A a K-algebra. A **matrix representation** of A over K is a K-algebra homomorphism $\boldsymbol{\delta}\colon A\to K^{n\times n}$ for some $n\in\mathbb{N}$. If V is a free K-module with a finite K-basis and $\mathrm{End}_K V$ is the K-algebra of all K-linear maps from V to V, a K-algebra homomorphism $\delta\colon A\to\mathrm{End}_K V$ is called a **representation** of A.

Remark 1.1.11 If $\boldsymbol{\delta}\colon A\to K^{n\times n}$ is a matrix representation of the K-algebra A then the free K-module $K^n:=K^{n\times 1}$ becomes an A-module by

$$a\cdot v:=\boldsymbol{\delta}(a)\,v\qquad\text{for}\qquad a\in A\ ,\ v\in K^n\,.$$

Conversely, if V is an A-module which is free as a K-module with K-basis $B:=(v_1,\dots,v_n)$, then we obtain a matrix representation $\boldsymbol{\delta}_B\colon A\to K^{n\times n}$ by defining

$$\boldsymbol{\delta}_B(a):=[\alpha_{ij}]\in K^{n\times n}\qquad\text{if}\qquad a\cdot v_j=\sum_{i=1}^{n}\alpha_{ij}v_i\qquad(a\in A\ ,\ 1\le j\le n)\,.$$

However, if V is an A-module which does not have a finite K-basis, V does not give rise to a matrix representation.

Notation: Let V and W be A-modules for a K-algebra A. Furthermore let $A' \subseteq A$, $a \in A$ and $U \subseteq V$.

(1) $U \leq_A V$ means "U is an A-submodule of V," that is, U is a subgroup of the additive group of V and $A \cdot U := \{a \cdot u \mid a \in A, u \in U\} \subseteq U$.

(2) $\ker_V A' := \{v \in V \mid A' \cdot v = 0\} \subseteq V$ and $\ker_V(a) := \ker_V\{a\}$.

(3) $\operatorname{ann}_A U := \{a \in A \mid a \cdot U = 0\} \leq_A {}_AA$, a left ideal in A.

(4)

$$\operatorname{Hom}_A(V,W) := \{\varphi\colon V \to W \mid \varphi(a \cdot v_1 + v_2) = a \cdot \varphi(v_1) + \varphi(v_2), \text{ for all } a \in A \text{ and } v_1, v_2 \in V\}$$

is the K-**module of all** A-**homomorphisms of** V **to** W.

(5) $\operatorname{End}_A V := \operatorname{Hom}_A(V,V)$ is the A-endomorphism ring of V with identity id_V, the identity map of V, and multiplication $\varphi\psi := \varphi \circ \psi$ for $\varphi, \psi \in \operatorname{End}_A V$, the composition of φ and ψ.

(6) $\delta_V\colon A \to \operatorname{End}_K V$ is defined by $\delta_V(a)(v) := a \cdot v$ for $a \in A$, $v \in V$. This is obviously a K-algebra homomorphism with kernel $\operatorname{ann}_A V$. If V has a finite K-basis, δ_V is called **the representation of** A **afforded by** V.

Observe that by Remark 1.1.5 A-modules V, W are also K-modules and it is clear that $\operatorname{Hom}_A(V,W) \subseteq \operatorname{Hom}_K(V,W)$. Remark 1.1.5 also shows that $E := \operatorname{End}_A V$ is a K-algebra with $\lambda_E(\alpha)\colon v \mapsto \alpha v$. The following is an obvious consequence from the definitions.

Lemma 1.1.12 *Let A be a K-algebra and V an A-module with $E := \operatorname{End}_A V$. Then V can be considered as an E-module via $\varphi \cdot v := \varphi(v)$ for $\varphi \in E$, $v \in V$, and*

(a) $\ker_V A' \leq_E V$ *for* $A' \subseteq A$,

(b) *if $U \leq_A V$, then $\operatorname{ann}_A U$ is a two-sided ideal in A, in symbols:* $\operatorname{ann}_A U \trianglelefteq A$.

Definition 1.1.13 Let A be a K-algebra and V an A-module with submodules $V_1, \ldots, V_n$. Then V is called the **direct sum** of $V_1, \ldots, V_n$, written

$$V = \bigoplus_{i=1}^{n} V_i = V_1 \oplus \cdots \oplus V_n \tag{1.2}$$

if every $v \in V$ can be written uniquely in the form $v = v_1 + \cdots + v_n$ with $v_i \in V_i$. For $1 \leq i \leq n$ define $\pi_i\colon V \to V_i$, $v \mapsto v_i$ and $\iota_i\colon V_i \to V$, $v_i \mapsto v_i$. Then $\pi_i \in \operatorname{Hom}_A(V, V_i)$, $\iota_i \in \operatorname{Hom}_A(V_i, V)$ are called the projections and injections corresponding to the decomposition (1.2). They satisfy

$$\mathrm{id}_V = \sum_{i=1}^{n} \iota_i \circ \pi_i \quad \text{and} \quad \pi_i \circ \iota_j = \begin{cases} 0 & \text{for } i \neq j, \\ \mathrm{id}_{V_i} & \text{for } i = j. \end{cases} \tag{1.3}$$

Remark 1.1.14 V in (1.2) is sometimes called the "internal direct sum" of the submodules $V_1, \dots, V_n$ in contrast to the "external direct sum," which can be defined for arbitrary (not necessarily distinct) A-modules $V_1, \dots, V_n$ as follows. Put $\hat{V} := V_1 \hat{\oplus} \cdots \hat{\oplus} V_n := \{(v_1, \dots, v_n) \mid v_i \in V_i\} = \{(v_i)_{i=1}^n \mid v_i \in V_i\}$ and

$$(v_i)_{i=1}^n + (v_i')_{i=1}^n := (v_i + v_i')_{i=1}^n \qquad \text{and} \qquad a \cdot (v_i)_{i=1}^n := (a \cdot v_i)_{i=1}^n.$$

Then $\hat{V}$ is an A-module and we have the obvious embeddings $\hat{\iota}_i \in \mathrm{Hom}_A(V_i, \hat{V})$ so that $V_1 \hat{\oplus} \cdots \hat{\oplus} V_n = \hat{\iota}_1(V_1) \oplus \cdots \oplus \hat{\iota}_n(V_n)$. We will identify V_i with $\hat{\iota}_i(V_i)$ and use only $\oplus$, leaving it to the reader to decide from the context whether the internal or external direct sum is meant.

Theorem 1.1.15 *Let A be a K-algebra and let $V := \bigoplus_{i=1}^m V_i$, $W := \bigoplus_{j=1}^n W_j$ be A-modules. Then*

$$\mathrm{Hom}_A(V, W) \cong_K \bigoplus_{i=1}^m \bigoplus_{j=1}^n \mathrm{Hom}_A(V_i, W_j),$$

where "$\cong_K$" means being isomorphic as K-modules.

Proof. Let $(\pi_i, \iota_i)_{i=1}^m$ and $(\pi_j', \iota_j')_{j=1}^n$ be the families of projections and injections corresponding to $V := \bigoplus_{j=1}^m V_i$ and $W := \bigoplus_{i=1}^n W_j$, respectively. Using (1.3) it is easily checked that

$$\mathrm{Hom}_A(V, W) \to \bigoplus_{i=1}^m \bigoplus_{j=1}^n \mathrm{Hom}_A(V_i, W_j)\ ,\ \varphi \mapsto (\pi_j' \circ \varphi \circ \iota_i)_{1 \le i \le m, 1 \le j \le n}$$

gives the desired K-isomorphism. □

Definition 1.1.16 Let V be a K-module with a finite K-basis. A representation $\delta\colon G \to \mathrm{GL}(V)$ of a group G over K or a representation $\delta_V\colon A \to \mathrm{End}_K V$ of a K-algebra A is called **irreducible** if the corresponding KG-module or A-module V is **simple**, i.e. has exactly two submodules, namely $\{0\}$ and V, and **reducible** otherwise. (So the zero module $\{0\}$ is by definition not simple.) Often simple modules are also called irreducible.

Likewise δ or δ_V as above is called **indecomposable** if the corresponding module is indecomposable (i.e. cannot be written as a direct sum of two non-trivial submodules) and is called **decomposable** otherwise.

If K is a field, we see that δ is reducible if and only if a basis $B = (v_1, \dots, v_n)$ of V can be found such that

$$[\delta(g)]_B = \left[\begin{array}{c|c} \boldsymbol{\delta}_1(g) & C(g) \\ \hline \mathbf{0} & \boldsymbol{\delta}_2(g) \end{array}\right] \qquad \text{for all } \ g \in G \tag{1.4}$$

with square matrices $\boldsymbol{\delta}_1(g) \in K^{m \times m}$, $\boldsymbol{\delta}_2(g) \in K^{(n-m)\times(n-m)}$ and $C(g) \in K^{m \times (n-m)}$ for some $0 < m < n = \dim_K V$. Here

$$\boldsymbol{\delta}_1\colon g \mapsto \boldsymbol{\delta}_1(g) \qquad \text{and} \qquad \boldsymbol{\delta}_2\colon g \mapsto \boldsymbol{\delta}_2(g)$$

are matrix representations of G afforded by the submodule $W = \langle v_1, \ldots, v_m \rangle_K$, the K-span of $v_1, \ldots, v_n$, with respect to the basis $(v_1, \ldots, v_m)$ and the factor module V/W with respect to the basis $(v_{m+1}+W, \ldots, v_n+W)$. A representation $\delta\colon G \to \mathrm{GL}(V)$ over a field K is decomposable if and only if a basis $B = (v_1, \ldots, v_n)$ can be found such that

$$[\delta(g)]_B = \left[\begin{array}{c|c} \boldsymbol{\delta}_1(g) & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\delta}_2(g) \end{array}\right] \quad \text{for all} \quad g \in G.$$

In this case W, as defined above, has a complement $W' = \langle v_{m+1}, \ldots, v_n \rangle_K$ which is also a KG-module and $\boldsymbol{\delta}_2$ is also a matrix representation of G afforded by W' (with respect to the basis $(v_{m+1}, \ldots, v_n)$).

Example 1.1.17 If V is any K-vector space we get for any group G a "trivial" representation $\delta\colon G \to \mathrm{GL}(V)\ ,\ g \mapsto \mathrm{id}_V$. This representation is irreducible if and only if $\dim_K V = 1$. ♦

Definition 1.1.18 If V is a KG-module then

$$\mathrm{Inv}_G(V) := \{v \in V \mid g \cdot v = v \quad \text{for all} \quad g \in G\}$$

is called the submodule of **G-invariants** of V. We also define

$$\mathrm{Inv}^G(V) := \langle (g-1)v \mid g \in G\ ,\ v \in V \rangle_K.$$

The notations $\mathbf{C}_V(G) := \mathrm{Inv}_G(V)$ and $[G, V] := \mathrm{Inv}^G(V)$ are also in use.

Obviously $\mathrm{Inv}_G(V)$ and $\mathrm{Inv}^G(V)$ are KG-modules, the largest submodule with trivial action of G and the smallest submodule W of V such that G acts trivially on V/W.

Example 1.1.19 The (left) regular KG-module ${}_{KG}KG$ for a finite group G leads to the so-called (left) **regular representation** $\rho_G\colon G \to \mathrm{GL}(KG)$ of G. It is easily seen that

$$\mathrm{Inv}_G(KG) = K \cdot \sum_{g \in G} g \qquad \text{and} \qquad \mathrm{Inv}^G(KG) = \{\sum_{g \in G} \alpha_g g \mid \sum_{g \in G} \alpha_g = 0\}.$$

In particular, the regular representation ρ_G is always reducible if G is not the trivial group. ♦

Definition 1.1.20 If A is a ring and $\varphi\colon A' \to A$ is a ring homomorphism, then any A-module V can be turned into an A'-module by defining

$$a' \cdot v := \varphi(a')\, v \qquad \text{for} \quad a' \in A'\ ,\ v \in V.$$

This A'-module will be denoted by $\mathrm{Inf}_\varphi V$ and is called the **inflated module**. If $\varphi\colon A' \to A$ is a homomorphism of K-algebras and V affords the representation δ of A then $\mathrm{Inf}_\varphi V$ affords the representation $\delta \circ \varphi$ of A', called the **inflated representation**. The same concept applies to group representations: if $\varphi\colon G_1 \to G$ is a group homomorphism and $\delta\colon G \to \mathrm{GL}(V)$ is a representation then $\delta \circ \varphi$ is the inflated group representation.

If A, A' are K-algebras, one should note that the K-module structures of V and $\mathrm{Inf}_\varphi V$ may be different, unless φ is a homomorphism of K-algebras; see Exercise 1.1.12 for an example.

Remark 1.1.21 Obviously, if $\mathrm{Inf}_\varphi V$ is simple (indecomposable) then V itself must be simple (resp. indecomposable); the converse holds too, provided that φ is surjective. Also, if V_1, V_2 are A-modules, then

$$\mathrm{Hom}_A(V_1, V_2) \subseteq \mathrm{Hom}_{A'}(\mathrm{Inf}_\varphi V_1\,,\, \mathrm{Inf}_\varphi V_2)$$

with equality, if φ is surjective.

The special and simple case that φ is an embedding needs special attention, since it is used so frequently. If A' is a subalgebra of A with embedding $\varphi\colon A' \to A$ and V is an A-module as before, then we write $V_{A'} := \mathrm{Inf}_\varphi V$. It is just the module obtained by restricting the action. In particular, if H is a subgroup of a group G and V is a KG-module we have an embedding of K-algebras $\varphi\colon KH \to KG$, and we denote the KH-module $\mathrm{Inf}_\varphi V$ by V_H.

Remark 1.1.22 If A is a K-algebra and V is an A-module which affords a matrix representation $\boldsymbol{\delta}\colon A \to K^{n\times n}$ with respect to some K-basis of V, then

$$V \cong_A \mathrm{Inf}_{\boldsymbol{\delta}} K^n,$$

where K^n is considered as a $K^{n\times n}$-module in the natural way.

In the following two examples we show that the representation theory of cyclic groups and, more generally, cyclic algebras over a field K is quite simple. In some important cases we will write down explicitly all irreducible and indecomposable representations up to equivalence.

Example 1.1.23 (Cyclic groups) If $\mathrm{C}_m = \langle g\rangle$ is a cyclic group of order m (with generator g), giving a matrix representation $\boldsymbol{\delta}$ of C_m of degree n over a field K amounts to giving a matrix $a := \boldsymbol{\delta}(g) \in K^{n\times n}$ with $a^m = \mathbf{I}_n$, where $\mathbf{I}_n$ denotes the identity matrix in degree n, or, to put it otherwise, to giving a matrix $a \in K^{n\times n}$ with minimal polynomial μ_a dividing $X^m - 1$ in the polynomial ring $K[X]$. Similar matrices define equivalent representations and vice versa. It is well known that a matrix a is diagonalizable in $K^{n\times n}$ (that is, similar to a diagonal matrix) if and only if the minimal polynomial μ_a decomposes into linear factors in $K[X]$ without multiple roots (see [153], theorem 8.11, p. 166). The polynomial $X^m - 1$ has no multiple roots if and only if $\mathrm{char}\, K \nmid m$. Thus it follows that if K is a field of characteristic not dividing m containing a primitive mth root ζ_m of unity, then any matrix representation of C_m is equivalent to one that maps the generator g to a diagonal matrix with mth roots of unity on the diagonal. In particular every irreducible matrix representation of C_m over such a field K (as for instance $\mathbb{C}$) is of degree one and of the form

$$g \mapsto \zeta_m^i \qquad \text{for} \quad i \in \{0, \ldots, m-1\}.$$

Also, every indecomposable representation is irreducible in this case.

On the other hand, if $\operatorname{char} K = p$, a prime, $m = p^s$ and $\boldsymbol{\delta}\colon \mathrm{C}_m \to \mathrm{GL}_n(K)$ is a matrix representation, then $\boldsymbol{\delta}(g)$ has a minimal polynomial dividing $(X-1)^m$, and hence $\boldsymbol{\delta}$ is equivalent to the representation given by the Jordan normal form of $\boldsymbol{\delta}(g)$ (see [153], theorem 8.6, p. 159):

$$\boldsymbol{\delta}\colon g \mapsto \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ \epsilon_1 & 1 & \ddots & \ddots & 0 \\ 0 & \epsilon_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \epsilon_{n-1} & 1 \end{bmatrix}$$

with $\epsilon_i \in \{0,1\}$, which is indecomposable if and only if all ϵ_i equal unity. Hence C_m has for $j = 1, \dots, m$ up to equivalence exactly one indecomposable representation of degree j and the trivial representation is the only irreducible one.

The structure of the group algebra of a cyclic group C_m of order m can also be easily described: keeping the notation as above we have an algebra epimorphism $K[X] \to K\,\mathrm{C}_m\ ,\ X \mapsto g$, with kernel the principal ideal $(X^m-1) \trianglelefteq K[X]$. Thus

$$K\,\mathrm{C}_m \cong K[X]/(X^m-1).$$

By the isomorphism theorem for rings the ideals of $K\,\mathrm{C}_m$ correspond bijectively to those ideals of $K[X]$ which contain (X^m-1), thus to the monic divisors of $X^m-1 \in K[X]$. Hence the poset of ideals of $K\,\mathrm{C}_m$ (with inclusions) is anti-isomorphic to the poset of monic divisors of X^m-1 (with divisibility as order relation). If $\operatorname{char} K \nmid m$ then $X^m-1 = \prod_{i=1}^r f_i$ with pairwise distinct monic irreducible polynomials $f_i \in K[X]$, and

$$K\,\mathrm{C}_m \cong K[X]/(f_1) \oplus \dots \oplus K[X]/(f_r)$$

is a direct sum of fields with the ith projection being given by $g \mapsto X + (f_i)$. In particular, if in addition K contains a primitive mth root of unity ζ_m then we may choose $f_i := X - \zeta_m^i \quad (1 \le i \le r = m)$, and it follows that

$$K\,\mathrm{C}_m \cong \underbrace{K \oplus \dots \oplus K}_{m}.$$

The m distinct projections are exactly the m irreducible representations of $K\,\mathrm{C}_m$ over K, the ith one yielding the representation defined by $g \mapsto \zeta_m^i$.

On the other hand, if $\operatorname{char} K = p > 0$ and $m = p^s$, the poset of ideals of $K\,\mathrm{C}_m$ is isomorphic to the poset of monic divisors of $X^m-1 = (X-1)^m$ and $K\,\mathrm{C}_m$ has exactly $m+1$ ideals I_i, all appearing in the unique composition series

$$K\,\mathrm{C}_m = I_m \rhd I_{m-1} \rhd \dots \rhd I_0 = \{0\}$$

with $\dim_K I_i = i$. ♦

Example 1.1.24 (Cyclic algebras) We generalize Example 1.1.23 and let A be a cyclic K-algebra over a field K, i.e. an algebra having one algebra generator, say a. So $A = K[a]$ is a homomorphic image of the polynomial ring $K[X]$ over K in one indeterminate X. Hence $A \cong K[X]/(f)$ for some $f \in K[X]$. If V is an A-module (affording the representation δ_V) then V can also be considered as a $K[X]$-module (see Remark 1.1.21) with $f \cdot v = 0$ for all $v \in V$ and, if $\dim_K V < \infty$, we can invoke the theorem on finitely generated modules over principal ideal domains (see [110], theorem III.7.5, p. 149) to conclude that

$$V \cong_{K[X]} K[X]/(q_1) \oplus \cdots \oplus K[X]/(q_m),$$

where $q_1, \ldots, q_m$ are powers of monic irreducible polynomials. The q_i are uniquely determined up to the ordering and are divisors of f. They are usually called the **elementary divisors** of $\delta_V(a)$ (see [153], p. 135). The modules $V_{q_i} := K[X]/(q_i)$ are annihilated by f and can thus be considered as A-modules, and the generating element a acts on V_{q_i} as X does; this means that we can find a K-basis B of V such that

$$[\delta_V(a)]_B = \begin{bmatrix} M_{q_1} & & 0 \\ & \ddots & \\ 0 & & M_{q_m} \end{bmatrix},$$

where M_q is the companion matrix (see [153], p. 146) of the monic polynomial q; i.e. if $q = \sum_{i=0}^{d} \alpha_i X^i$ with $\alpha_d = 1$ then

$$M_q = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & & & -\alpha_1 \\ 0 & \ddots & \ddots & & \vdots \\ . & \cdots & \ddots & 0 & . \\ 0 & 0 & & 1 & -\alpha_{d-1} \end{bmatrix} \in K^{d \times d}.$$

Observe that the characteristic polynomial of $\delta_V(a)$ is exactly $\prod_{i=1}^{m} q_i$. The poset of submodules of V_{q_i} is isomorphic to the poset of divisors of q_i in $K[X]$. In particular V_{q_i} is always an indecomposable A-module and it is simple if and only if q_i is irreducible in $K[X]$. If $p \in K[X]$ irreducible of degree s and $q = p^e$ then by Exercise 1.1.11 M_q is similar to

$$M'_q := \begin{bmatrix} M_p & 0 & \ldots & 0 \\ E & M_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & E & M_p \end{bmatrix}, \text{ with } E = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ddots & & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & & 0 \end{bmatrix} \in K^{s \times s}. \quad (1.5)$$

For later use (in Lemma 1.3.9) we note that

$$\dim_K \ker_V(p_i(a)) \geq \deg p \quad (1.6)$$

because M_p has minimal polynomial p. If equality holds in (1.6) then there is exactly one elementary divisor q_i which is a power of p. ♦