Matroids: A Geometric Introduction

Matroid theory is a vibrant area of research that provides a unified way to understand graph theory, linear algebra and combinatorics via finite geometry. This book provides the first comprehensive introduction to the field, which will appeal to undergraduate students and to any mathematician interested in the geometric approach to matroids.

Written in a friendly, fun-to-read style and developed from the authors' own undergraduate courses, the book is ideal for students. Beginning with a basic introduction to matroids, the book quickly familiarizes the reader with the breadth of the subject, and specific examples are used to illustrate the theory and to help students see matroids as more than just generalizations of graphs. Over 300 exercises are included, with many hints so students can test their understanding of the material covered. The authors have also included several projects and open-ended research problems for independent study.

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This book is dedicated to Thomas Brylawski (1944–2008) who taught both authors matroid theory. His enthusiasm, energy and humor were inspiring to his students, colleagues and friends.

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Preface

Matroids – a quick prehistory

Matroid theory is an active area of mathematics that uses ideas from abstract and linear algebra, geometry, combinatorics and graph theory. The study of matroids thus offers students a unique opportunity to synthesize several different areas within mathematics typically studied at the undergraduate level. Furthermore, matroids are an active area of research; *Mathematical Reviews* lists some 2000 publications with the word “matroid” in the title, with more than a third of these appearing in the last decade.

Why have we written this book? Our motivation is direct: There is no comprehensive text written for undergraduates on this topic. There are several more advanced treatments of the subject, suitable for graduate students or researchers, but most of these are difficult for undergraduates to read. To paraphrase an old joke, this text seeks to fill this “much-needed gap.”

This text introduces matroids by emphasizing geometry, focusing especially on geometric (*affine*) dependence. Interpreting this approach for finite subsets of a vector space, points in Euclidean space or the edges of a graph gives a matroid spin to linear algebra, discrete geometry and graph theory. We believe the geometric approach, which both authors learned from their common Ph.D. advisor, Thomas Brylawski, to be the most natural, useful and powerful in understanding the subject.

The common thread that ties the various classes of matroids together is the abstract notion of independence. This unifying idea is due to Hassler Whitney, who defined matroids in his foundational paper [42] in 1935. The field developed slowly in the 1940s and 1950s, attracting the attention of Garrett Birkhoff [3], who studied the flats of a matroid from a lattice-theoretic viewpoint, Saunders MacLane [23], who related matroids to projective geometry, and Richard Rado [27], who made important connections between the transversals of a bipartite graph and matroids. Most significantly, William Tutte [35] anticipated much of the modern approach to the field in 1958, where he characterized binary and regular matroids in terms of *excluded minors*. Finding excluded minors for classes of matroids remains one of the deepest and most vibrant areas of the field today.

Matroids can be defined in a surprisingly large number of *cryptomorphic* (= "secretly isomorphic") ways, and using different characterizations can be quite useful for different kinds of problems (it can also be a bit daunting for the beginner). The fact that there are so many different ways to define a matroid is a striking feature of the subject. Gian-Carlo Rota, a mathematician who taught several of the top researchers in the field, remarked that "It is as if one were to condense all trends of present-day mathematics onto a single finite structure, a feat that anyone would *a priori* deem impossible, were it not for the mere fact that matroids do exist."

Using this text

Since matroids impinge on several subjects studied at the undergraduate level, we believe this book could serve as an ideal text for a capstone course. Such a course might incorporate and unify abstract algebra, linear algebra, combinatorics and geometry. The text could also serve as a special topics course in linear algebra, geometry or combinatorics. In addition, the text could form the basis of an independent study course for undergraduates or beginning graduate students.

Although the study of any mathematical subject requires the reader to understand the conventions and details of mathematical proofs, we have tried to make this process as gentle as possible. Proofs are often accompanied by examples that illustrate the main ideas in the proof. Examples abound throughout, and the reader is given ample opportunity to understand the theory by working some 300 exercises.

While no specific background is assumed, a reader who has studied some linear algebra and understands the basics of mathematical reasoning should be ready to read this book. Some exposure to abstract algebra, combinatorics or graph theory might be useful, but is not a requirement. We have included introductory material on graphs, matrices and linear algebra, finite fields, and finite geometry. Some of this material appears as a subsection of a chapter, some as a section, and we devote an entire chapter (Chapter 5) to finite geometry, a topic not often taught to undergraduates. Depending on the background and preparation of a class, an instructor might choose to review material by lecture, by student presentations, or by independent study. Of course, any instructor may opt to select chapters to match the students' backgrounds.

A quick note about style: We have tried to make this book somewhat "chatty." This means we have included some jokes and lots of ancillary material on related areas of mathematics. Writing a mathematics text in a conversational style is a tricky business; if the jokes are overdone (or too corny, or not corny enough, or . . .), then this may turn readers away.

On the other hand, both authors have taught enough undergraduates to appreciate a text that shows some of its author's personality. We hope the reader will be tolerant with this approach, and understand what we've tried to achieve.

Note to instructors

Chapter 1 is designed to pique the student's interest through lots of examples. In this introductory material, we try very hard to develop some geometric intuition, and we do so without proofs. We recommend instructors spend enough time on this chapter to develop this intuition in their students.

Chapter 2 is long,[1] and it gives many of the equivalent (cryptomorphic) definitions of a matroid; it is an important chapter and one that the students typically find challenging. We recommend a thorough treatment (complete with proofs) of Sections 2.1–2.3. Moving quickly through the remaining sections of the chapter is recommended, with instructors choosing how much or how little to cover. While it is amazing (to us) that matroids can be defined in so many different ways, most of our students have hidden their amazement very well. Consequently, it's easy to get bogged down with this material. Moving on to matroid operations (Chapter 3) when your students have had their fill of cryptomorphisms is a wise approach.

Chapters 3, 4, 5 and 6 are the heart of the text. Chapter 3 introduces matroid operations and is needed for all subsequent chapters. This includes the fundamental operations of deletion, contraction and duality, all of which are of central importance. We have included applications to graphs and matrices here – the entire notion of duality is presented as a generalization of duals for planar graphs. Chapter 4 treats graphs carefully, providing matroid proofs for many theorems of graph theory. We also make a general argument for generalization: proving a theorem about matroids gives you, *gratis*, theorems about graphs, vector spaces, and all the other classes of matroids we care about.

Chapter 5 gives an extensive treatment of affine and projective geometry, and it also includes a quick and dirty introduction to finite fields and vector spaces. This is beautiful, classical material, and it has somehow slipped through the cracks of modern undergraduate education. Chapter 6 builds on the finite geometry of Chapter 5 with a treatment of matroid representation questions. This chapter gives careful proofs of non-representability for several classical examples, including the Desargues' and Pappus' configurations.

Chapters 7, 8 and 9 include various special topics students often find appealing. Chapter 7 treats two important classes of matroids in some depth: transversal matroids (based on matchings in bipartite graphs)

[1] Paraphrasing Pascal, we would have written a shorter chapter if we had had more time.

and hyperplane arrangements in Euclidean space. An instructor could choose to cover either of these topics, or both (or neither). Chapter 8 is the deepest chapter of this text, and it treats the problems of excluded minors. We have attempted to give as much example-based proof as possible here, and this is especially true in our proof of Tutte's Theorem on excluded minors for binary matroids. Chapter 9 develops some of the theory of the Tutte polynomial, a two-variable matroid invariant that is also a very active area of current research.

It is not possible in such a text to be comprehensive, of course. The topics we have selected reflect our taste (and the tastes of our students), but the common theme is the geometric approach to the subject. The field has grown tremendously in the past dozen years, with major advances from a variety of researchers. But there are also opportunities for beginners to make contributions to the subject. We have included several projects that could serve as jumping off points for ambitious students.

A typical one-semester course might include the introductory material in Chapter 1, the first three sections of Chapter 2 and all of Chapter 3, followed by three or four additional chapters, ending perhaps with student projects. For instance, one could follow the material in the first three chapters by covering Chapters 4, 5 and 9 (for a gentle introduction), Chapters 4, 5, 6 and 8 (for a course focusing on matroid operations and minors), or Chapters 4, 5, 6 and 7 (for a thorough treatment of matroid classes). The flowchart below shows the interdependencies of the chapters.

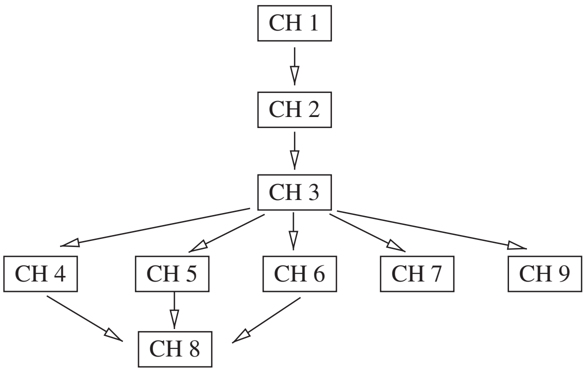

Figure 1. Interdependence between chapters.

A note to students

This book is meant to be read actively! That means having paper and pencil out while you're reading, with an example or two in mind. Then try your example on each new theorem, definition or concept you encounter in the text. While the theory can be rather abstract, the concrete examples should help fix the ideas in your mind. We've included lots of examples in the text to help you get used to this style of reading.

The exercises are meant to be worked through carefully. There are plenty of computational problems, but there are also a large number of problems where we ask you to show or prove something. Some of these are routine proofs that follow closely a proof in the text, but many of them require some new idea to complete. We have given extensive hints on the harder problems, and some of the more important exercises appear later in the text as propositions, with proofs.

A note about the jokes. The footnotes contain lots of remarks that might be uttered in a lecture, but usually don't survive the editing process in writing a book. If you like these comments, feel free to contact the authors to express your appreciation; if you don't, you should not necessarily feel compelled to tell us.

A general comment on reading mathematics: draw lots of pictures and don't get bogged down in terminology, definitions and details. Most professional mathematicians read technical material in stages, first skimming them for the main ideas, then going back and working through the details at one level (still skipping some technical details), then, finally, going back again to get all of the details down. Most of the time, there is a nicely chosen example (usually a picture) being updated as they read.

Our goal in writing this book is to show you both the beauty of matroids as well as the interconnectedness of mathematics. We use ideas from many different branches of mathematics in this book, and (we hope) this approach will strengthen your overall understanding of abstract mathematics. We hope you will learn much, enjoy the book, and laugh at our jokes.

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Finally, Thomas Brylawski's (1944–2008) spirit guided us throughout the project.