Cambridge University Press
978-0-521-76724-8 - Matroids: A Geometric Introduction
Gary Gordon and Jennifer McNulty
Excerpt
More information

1

A tour of matroids

1.1 Motivation

Matroids were introduced by Hassler Whitney, in a seminal[1] paper [42] that anticipated much of the early development of the subject.[2] Since its birth, the theory of matroids has undergone enormous growth, and it remains one of the most active research areas in mathematics. We'll give you more historical nuggets about the development of matroid theory along the way (often in footnotes), but not now.

Right now, as a tease, here's a question for you: How are each of the things[3] in Figure 1.1 related to each other, and how are they related to the following matrix?

$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ B = & \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right]. \end{array}$$

This chapter should help you to answer this open-ended question, with matroids playing the unifying role. In particular, you should be able to find each of these pictures – and lots of matrices – in this and subsequent chapters.

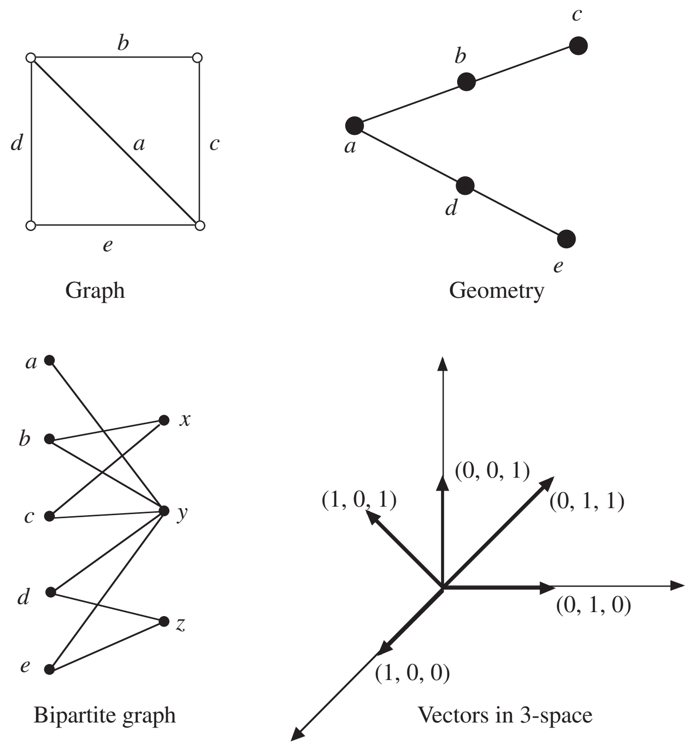
Typically, when you study mathematics, especially in the undergraduate curriculum, you study topics separately, in individual courses like *linear algebra, abstract algebra, discrete math, combinatorics, geometry, graph theory*, and so on. While it's not illegal to think about a topic from *abstract algebra* (algebraically closed fields, say) while you're

[1] There will be plenty of footnotes in the text. You can ignore them, or just glance at them to decide if reading them is worth the effort. There is no reason we footnoted this word – it's just to get you used to looking down.

[2] Hassler Whitney (1907–1989) was an outstanding mathematician with wide interests; his fundamental work in algebraic topology, differential geometry and differential topology earned him the Wolf Prize in 1983. He invented matroid theory in the 1930s after working in graph theory. Whitney was an avid and accomplished rock climber, and his grandson wrote about his famous unprotected ascent in 1929 of the Cannon Cliff in NH, now named the Whitney–Gilman Ridge.

[3] This is not a technical term.

Figure 1.1. Four pictures.



sitting in your *complex analysis* course, the textbooks for these courses and the way most undergraduate courses are designed do not encourage you to do so.

Our main point is this: much of the beauty and the richness of mathematics comes from the many connections between the various branches of mathematics. We believe the study of matroids is especially well suited for this purpose.[4] Figure 1.1 shows four objects that represent the same matroid-dependence structure: a graph, a point–line incidence geometry, a bipartite graph and an arrangement of vectors. Furthermore, they are all equivalent – as matroids – to the column dependences of a matrix, giving us a connection to linear algebra, as well.

Matroid theory uses linear and abstract algebra, graph theory, combinatorics and finite geometry.

You shouldn't understand any of the details yet – we haven't given any. This chapter is devoted to introducing you to matroids by giving you lots of examples as they appear in several different areas of mathematics.

[4] Well, of course we believe this – we've written a book about them.

After reading this chapter, you should be able to understand what a matroid is from the viewpoint of graphs, linear algebra and geometry.

This text emphasizes the geometric approach popularized by Gian-Carlo Rota.[5] We learned Rota's approach from his student Thomas Brylawski, the Ph.D. advisor for both authors.[6]

Rota used the term "combinatorial pregeometry" instead of the term matroid, but this is sufficiently awkward to have not caught on. Rota had very strong feelings about terminology, believing the word "matroid" to be "ineffably cacophonic." In 1971, he and Kelly wrote:

> Several other terms have been used in place of geometry, by the successive discoverers of the notion; stylistically, these range from the pathetic to the grotesque. The only surviving one is "matroid," still used in pockets of the tradition-bound British Commonwealth. [19]

1.2 Introduction to matroids

We will tell you what a matroid is very soon – we promise – but we begin with two examples.

Example 1.1. Let A be the following matrix.

$$A = \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array}.$$

We care about the four columns[7] $a = (1, 0)$, $b = (0, 1)$, $c = (1, 1)$ and $d = (1, 2)$. More to the point, we are interested in those subsets of columns that are *linearly independent* and those that are *linearly dependent*. (Remember that a set of vectors is *linearly dependent* if some non-trivial *linear combination*[8] of the vectors is the zero vector.)

Now in our matrix A, every pair of vectors forms a linearly independent (= not linearly dependent) subset of $\mathbb{R}^2$, but any subset of three of these vectors forms a linearly dependent set because the vectors all live in $\mathbb{R}^2$. (It is always true that when the number of vectors is larger than the dimension of your space, the vectors are linearly dependent.) Of course, the entire set of four vectors is linearly dependent.

How can we describe the linearly dependent subsets of $\{a, b, c, d\}$? Here's a surefire way that should appeal to the computer

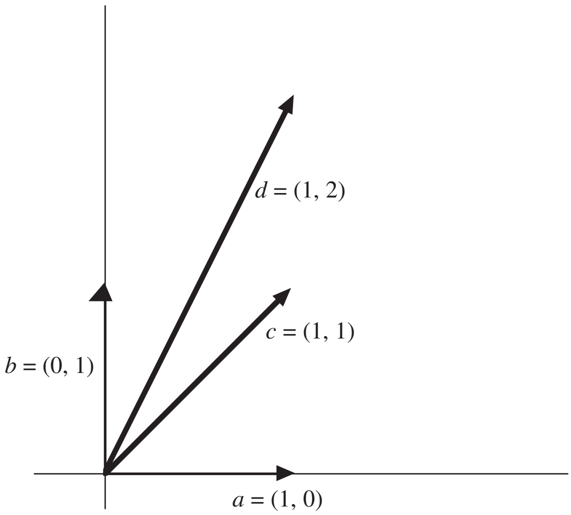
[5] Gian-Carlo Rota (1932–1999) was an eloquent mathematician and philosopher who worked in combinatorics, but also made deep contributions to invariant theory and analysis. He received the Steele Prize in 1988 for his paper *On the foundations of combinatorial theory* [29] which is credited as "... the single paper most responsible for the revolution that incorporated combinatorics into the mainstream of modern mathematics."

[6] This last paragraph really belongs in the Preface, but scientists have proven that nobody ever reads the Preface.

[7] For convenience, we will sometimes write column vectors horizontally, like so: $(1, 0)$.

[8] This is the problem with definitions – they rely on *other* definitions.

Figure 1.2. Four vectors.



programmer: list them all. In this case, it's easy: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$. You should convince yourself that this is not a good approach in general.

While there are many ways to "describe" these sets, we focus on a geometric way that will be central to the rest of this text (you might want to pay attention now). We will draw a picture that represents the linear dependence and independence of the subsets of the four columns from the example. The procedure has three easy steps.

Rank 2 matroid drawing procedure from a matrix

- Step 1: Draw the vectors in the plane. See Figure 1.2.
- Step 2: Draw a line in a "free" position – this means we want a line that is not parallel to any of our vectors. Now extend or shrink (and reverse, if necessary) each vector to see where it would hit this free line. See Figure 1.3.[9]
- Step 3: Finally, to get a picture of the column vector dependences corresponding to this matrix, just keep the line and discard the original vectors. See Figure 1.4 for a picture of the resulting "matroid," which just consists of four collinear points.

Here are two important things that you might notice.

(1) The length of a vector doesn't matter; for example, the picture in Figure 1.4 would be the same if we replaced (1, 1) by (2, 2).

[9] Hey – isn't this really two steps?

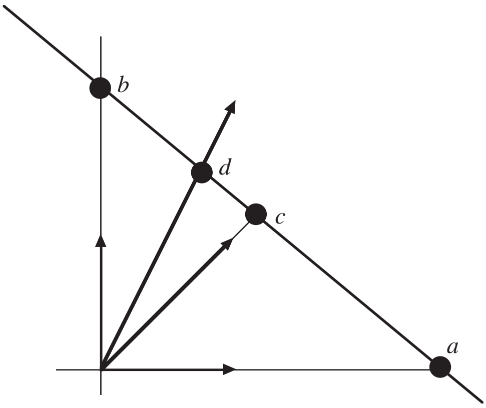


Figure 1.3. Our four vectors and a "free" line.

Figure 1.4. A picture of the matroid for the column dependences of the matrix A.

(2) We could replace a vector by its negative without changing our picture in Figure 1.4; for instance, replacing $(1, 2)$ by $(-1, -2)$ wouldn't change which subsets of vectors were dependent.

Example 1.2. Let's do another example. This time, let B be the following matrix:

$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ B = & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}. \end{array}$$

As before, we wish to draw a picture that represents the column dependences in the matrix. This time, we'll think of this as projecting the vectors onto a free plane, so we use the same drawing procedure as before, substituting "plane" for "line" in step 2:

Rank 3 matroid drawing procedure from a matrix

- As before, draw the column vectors – or, better yet, make a three-dimensional model of the vectors using some nice model building kit (or, if no such kit is available, use toothpicks and gumdrops).
- Next, find a plane P that is "free" with respect to your set – that means a plane that is not parallel to any of your vectors. See Figure 1.5.
- Finally, extend or shrink each of your vectors (or their negatives) until the extended or shrunken vector meets your plane P. These points in the plane will be the picture of your column vector dependences. See Figure 1.6. Congratulations!

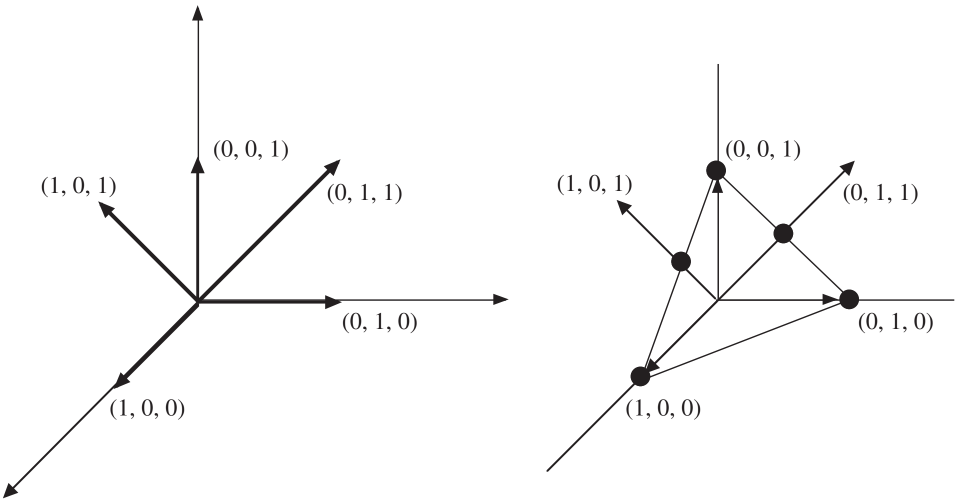
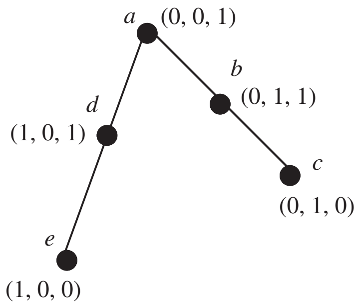


Figure 1.5. Projecting vectors in $\mathbb{R}^3$ onto the plane $x + y + z = 1$.

Figure 1.6. The matroid corresponding to the column vectors of the matrix B with the two-point lines omitted.



One more important comment about the picture in Figure 1.6 – by convention, we don't draw line segments connecting two points if they are the only two points on that line. For example, the points b and d form a two-point line, but we don't draw this line. The main reason for this is that no one else draws these lines,[10] but adding these lines would also increase the clutter in the picture (see Figure 1.7). There are four two-point lines in this matroid: $\{b, d\}$, $\{b, e\}$, $\{c, d\}$ and $\{c, e\}$. Remember – even though we haven't drawn two-point lines, they're still there.

Our drawing procedure amounts to a way of reducing dimension: in our first example, the rank of the matrix A was 2, but the corresponding picture – the four-point line – is a one-dimensional object. (Recall the *rank of a matrix* is the dimension of its row space or its column space – these two subspaces have the same dimension.) In the second example, the matrix rank for B is 3, but our matroid dependence picture was

[10] We know this sends a mixed message about peer pressure, but that's life.

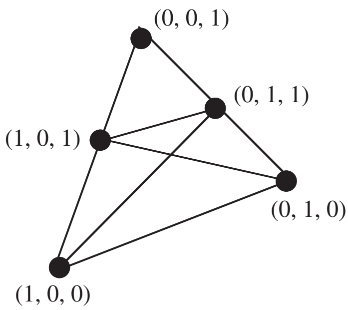


Figure 1.7. The matroid corresponding to the column vectors of the matrix B, with two-point lines drawn.

two-dimensional. So, in each case, we have "rank = dimension + 1" for the matroid dependence pictures we will draw.

It's now time for our first definition of a matroid[11] – you'll see several equivalent definitions in Chapter 2.

Definition 1.3. Let E be a finite set and let $\mathcal{I}$ be a family of subsets of E. Then the family $\mathcal{I}$ forms the *independent sets of a matroid M* if:

Matroid definition

(I1) $\mathcal{I} \neq \emptyset$;
(I2) if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$;
(I3) if $I, J \in \mathcal{I}$ with $|I| < |J|$, then there is some element $x \in J - I$ with $I \cup \{x\} \in \mathcal{I}$.

Non-triviality
Closed under subsets
Augmentation

The set E is called the *ground set* of the matroid. In our example, E was a set of vectors, but in another important example, E will be the edges of a graph. The *rank* of a matroid, which is written $r(M)$, is just the size of the largest independent set. The matroid associated to the matrix A in Example 1.1 has rank 2, and the matroid associated to B in Example 1.2 has rank 3. Most of the examples in this chapter have rank 3. Also, the matroid rank equals the matrix rank – that seems fortuitous. Bet you didn't see that one coming.

Ground set
Rank
Matroid rank = matrix rank

This definition was first formally stated by Whitney [42]. He noticed that the independence properties (I1), (I2) and (I3) were enjoyed by linearly independent subsets of a *vector space*, and he wanted to understand how much (or how little) of the special features of vectors depend on the field of coefficients (more precisely, how much of linear algebra is independent of coordinates).

We'll prove that finite sets of vectors are examples of matroids in Theorem 6.1 in Chapter 6. In that chapter, we concentrate on the connections between matroids and matrices.

Theorem 6.1. *Let E be the columns of a matrix A with entries in a field $\mathbb{F}$, and let $\mathcal{I}$ be the collection of all subsets of E that are linearly independent. Then $M = (E, \mathcal{I})$ is a matroid, that is $\mathcal{I}$ satisfies the independent set axioms (I1), (I2) and (I3).*

Matrices give matroids

[11] Your job: Read the definition and turn it into sentences you can understand.

Definition 1.4. A matroid whose ground set E is a set of vectors is called a *representable matroid*.

Representable matroid

Not all matroids are representable, and we'll study these matroids in some detail in Chapter 6. For now, note that it is very easy to see why properties (I1) and (I2) are satisfied by subsets of linearly independent vectors. It's a bit more work to check that (I3) always holds – this is really a fact from linear algebra. We defer the proof of Theorem 6.1 to Chapter 6.

A not-matroid

Example 1.5. It helps to understand a definition by looking at examples where it is *not* satisfied. As an easy example, suppose $E = \{a, b, c, d\}$ and you are given the following subsets: $\emptyset, a, b, c, d, ab, cd$. If you don't mind, for the sake of brevity, we will write ab instead of $\{a, b\}$, cd instead of $\{c, d\}$, and so on.[12]

Could these subsets be the independent sets of some matroid? (Pause to think!) The answer is no. While the subsets satisfy (I1) and (I2), axiom (I3) is violated: c and ab both independent requires either ac or bc to be independent.

By the way, it's possible to add some new sets to $\mathcal{I}$ and satisfy (I3). Of course, we could simply add all subsets of E to $\mathcal{I}$, but this is overkill. You should check that adding the two subsets ad and bc to $\mathcal{I}$ works.

Uniform matroids

Example 1.6. This time, let $E = \{e_1, e_2, \ldots, e_n\}$. Let $k \leq n$ and define $\mathcal{I}$ to be all subsets of E with k or fewer elements. (For example, the matroid from Example 1.1 has this property for $n = 4$ and $k = 2$.) Then $\mathcal{I}$ satisfies (I1), (I2) and (I3). This is called the *uniform* matroid, and it's denoted $U_{k,n}$. You will see it frequently in this text; see Exercise 4.

Boolean algebra

The matroid $U_{n,n}$ is called the *Boolean algebra*, and we denote it by B_n. Every subset is independent, and this clearly satisfies (I1), (I2) and (I3).[13]

Before finishing this section, we give two more matrix examples. In particular, given the column vectors of a matrix, could we skip all the vector drawing and jump straight to the matroid picture somehow? Well, could we?

The answer is yes, if we're a tiny bit clever.[14] We use the following picture drawing rules to go directly from our matrix A to a matroid picture:

[12] You really have no choice here.

[13] George Boole (1815–1864) was the most famous logician of his day. Boole's daughter Alicia Boole Stott made important contributions to higher-dimensional geometry, for instance, proving there are precisely six regular solids in four dimensions, and constructing physical models of them.

[14] It's even easier if we're very clever.

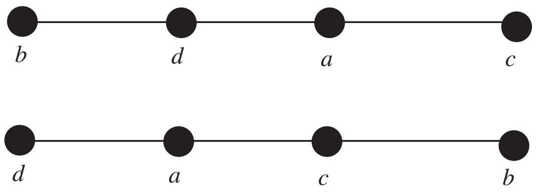


Figure 1.8. Two different labelings for the column dependences of A.

- each column vector will be represented by a point.
- if three vectors u, v and w are linearly dependent, then the corresponding three points will be collinear.

Only using these two rules frees us up a little. For instance, we can now represent the column dependences of

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ A = & \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

with either of the pictures in Figure 1.8. There are plenty[15] of other labelings of the four-point line that work.

Here's why the drawing procedure works in this example:

Three vectors are linearly dependent ⇔
the vectors are coplanar ⇔
the three corresponding points in our matroid picture are collinear.

This is easy to see for vectors in $\mathbb{R}^2$: two vectors in the plane are linearly independent as long as they point in different (not opposite) directions. But, if they point in different directions, then they meet our free line in distinct points, so they're independent in the matroid.

What if a pair of vectors is linearly dependent? Two dependent vectors will result in a pair of "multiple points" in the matroid, i.e., a dependent set of size 2. It is even possible for our matroid to have dependent *singletons*. Both of these pathologies occur in the next example.

Example 1.7. Let C be the following matrix.

$$\begin{array}{cc} & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ C = & \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & -4 & 0 \end{bmatrix}. \end{array}$$

Since C is a rank 3 matrix, we expect our matroid picture to be planar. Two features of this matrix we have not seen before deserve some attention.

- g, which corresponds to the zero vector, is a dependent set of size 1.
- The pair df is a dependent set of size 2.

15 4! We're excited about this!

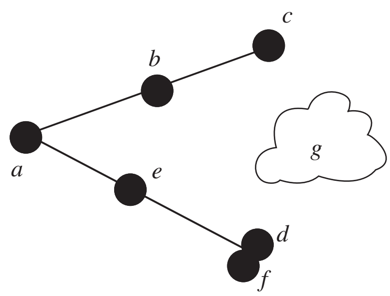


Figure 1.9. A picture of the matroid on the columns of C.

Other dependent sets should be more familiar. For instance, note that columns a, b and c are linearly dependent:

$$(1, 1, 0) + (0, -1, 1) - (1, 0, 1) = (0, 0, 0).$$

From the linear dependences in the matrix, we get the following dependence story in the matroid:

(1) g is a dependent set.
(2) df is a dependent set; all other pairs of points taken from the set $\{a, b, c, d, e, f\}$ are linearly independent.
(3) abc, ade and aef are dependent; all other triples of points taken from the set $\{a, b, c, d, e, f\}$ that don't contain both d and f are linearly independent.
(4) Any set of four or more points is dependent.

The picture in Figure 1.9 gives all of the dependence information that the matrix did (if we interpret four or more coplanar points as a dependent set). For example, the three points a, c and e correspond to a linearly independent set of vectors (since the three points are *not* collinear). Note the two three-point lines abc and ade correspond to the two linear dependences $1 \cdot a + 1 \cdot b - 1 \cdot c = 0$ and $1 \cdot a + 1 \cdot d - 1 \cdot e = 0$. As usual, we don't draw two-point lines, like ce or the line through b and the double point df.

How do we represent the double point df? And what is going on with that weird cloud-like object that seems to have swallowed g? Well, we[16] have a fundamental problem with trying to represent dependent sets of size 1 or 2 geometrically. For multiple points, this problem isn't too serious; we simply overlap our big black disks suggestively.

For the dependent set of size 1, we have a more fundamental problem. It's not really possible to draw this as a point in any reasonable way, so we indicate this by enclosing g in a cloud.[17] By the way, a dependent singleton is called a *loop*, and you'll see loops throughout the text. They are more important than you might guess at first glance.

Loop = dependent singleton

[16] Everyone, really – not just us.
[17] Did you know Montana is Big Sky country?